HRW Day 02 — The Torus Algebra and the Type D Equation

2026-10-24

Day 02 — The Torus Algebra and the Type D Equation

Course date: 2026-10-24. **Language:** English. **Target:** graduate qualifying examination. **Study allocation:** 4 h 45 min.

PDF attachment: [[HRW_Day02.pdf]]. **Source:** [[HRW_Day02_Source.tex]] (same folder). These are newly authored replacement notes for previously unavailable Day 02 files, not recovered copies of the original lectures.

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Reading assignment and theorem scope

Main paper: HRW23 §2.1–2.3, especially the algebra conventions and train-track definition; HRW22 §2 for grading context. Consult HRW23 and HRW22. These notes follow the **source's algebraic and geometric terminology**. A statement identified as a source theorem is not claimed to have been proved in full merely because an illustrative calculation is supplied. We use the 2023 main manuscript and the 2022 published companion; theorem numbers can differ in earlier preprints.

Core definitions and exact algebraic framework

> [!theorem] Source-grounded mathematical content A parametrized torus is encoded by an oriented pointed matched circle and two idempotents ι_0, ι_1. The algebra has six Reeb-chord elements $\rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}$. Its nonzero nonidempotent products are $\rho_1\rho_2 = \rho_{12}$, $\rho_2\rho_3 = \rho_{23}$, $\rho_1\rho_{23} = \rho_{123}$, and $\rho_{12}\rho_3 = \rho_{123}$. A type D structure is an idempotent-graded vector space V and a map $\delta^1 : V \to \mathcal{A} \otimes_{\mathcal{J}} V$ with $(\mu \otimes 1)(1 \otimes \delta^1)\delta^1 = 0$. The idempotent restrictions matter: $\rho_1, \rho_3, \rho_{123}$ run from ι_0 to ι_1, ρ_2 from ι_1 to ι_0, ρ_{12} from ι_0 to itself and ρ_{23} from ι_1 to itself. A graph edge is an algebra coefficient, not a plain differential arrow.

Context and motivation. Bordered Floer theory expresses gluing through a type A/type D pairing, while the immersed-curve reformulation turns the pairing into a geometric intersection problem on a once-punctured torus. The technical challenge is preserving the information of the coefficient algebra, local systems, and admissibility under graphical changes. A concrete computation must therefore be separated from a theorem asserting equivalence of whole modules.

Categories. The geometric model is a punctured **surface** with immersed curves and finite-dimensional local systems. It does not assert that every algebraically admissible curve system comes from a smooth three-manifold. All results here are over $\mathbb{F}_2$.

The main proof or proof checkpoint

[!proof] Mathematical argument and its status **Derived, complete algebra check.** If $\delta^1(x) = \rho_1 \otimes y$ and $\delta^1(y) = \rho_2 \otimes z$ with no other length-two path from x to z, then $\delta^2(x) = \rho_{12} \otimes z \neq 0$. Thus this directed graph is invalid. Cancellation is possible only when another two-step path produces exactly the same basis coefficient and target, since we work over $\mathbb{F}_2$. A second computation checks associativity: $(\rho_1\rho_2)\rho_3 = \rho_{12}\rho_3 = \rho_{123} = \rho_1\rho_{23} = \rho_1(\rho_2\rho_3)$. These are concrete checks, not a proof of every A_∞ identity.

What to defend orally. Identify the exact algebraic equation, explain why its hypotheses hold, and distinguish the complete calculation above from the source-dependent geometric normal-form or bigon argument. The latter requires the relevant diagrams of HRW23; the text here does not silently replace those diagrams with a heuristic.

Worked computation and interpretation

Let $u = (1, 0)$ and $v = (2, 5)$ be primitive slope vectors. Their geometric intersection number is $|\det(u, v)| = 5$. The idempotent structure of a bordered module is additional data: knowing this determinant does not specify a type D differential. In a reduced graph, a coefficient ρ_{12} starts and ends in the ι_0 summand; a purported ρ_2 edge starting at ι_0 violates idempotent compatibility even before testing $\delta^2 = 0$.

Second check (algebraic invariant). An invertible matrix A and a basis change P define isomorphic local systems precisely when the monodromies are conjugate, $A' = P^{-1}AP$. Similarity preserves characteristic and minimal polynomials, but agreement of those two polynomials alone need not imply similarity. This observation becomes decisive in the classification proof.

Geometric check. For primitive nonparallel slope vectors $u = (a, b)$ and $v = (c, d)$, $i(u, v) = |ad - bc|$. This formula counts geometric intersections of straight representatives; when the curves are parallel, or when local-system monodromy is nontrivial, the Floer homology cannot be read from the determinant alone.

Source-specific pitfalls

[!warning] Convention and hypothesis audit The algebra has zero internal differential, but the induced type D differential need not be zero. The type D equation involves **multiplication of chord coefficients**. In particular, a chain of two arrows is not automatically allowed. The algebra is over $\mathbb{F}_2$, so cancellation is parity-sensitive.

For any argument involving closed-manifold Floer homology, the gluing homeomorphism reverses boundary orientation. For the type A/type D pairing, the elliptic-involution correction and reflection convention must be tracked; HRW22's elliptic symmetry permits simplification for **actual manifold invariants**, not automatically for every abstract curve collection.

Written qualifying examination

Instructions. Attempt these six questions without notes; write complete algebraic proofs and explicitly label any diagrammatic proof sketch.

[!exercise] Problem 1 List all six chord generators and their idempotents.

[!exercise] Problem 2 Verify the four nonzero products of nonidempotent generators.

[!exercise] Problem 3 Show why a lone ρ_1-then-ρ_2 path violates the type D equation.

[!exercise] Problem 4 Give the algebraic cancellation condition for two length-two paths.

[!exercise] Problem 5 Check the associativity identity producing ρ_{123}.

[!exercise] Problem 6 Explain why an unoriented geometric curve does not determine a type D module without local-system data.

Solutions and hints

[!proof] Problem 1: solution strategy / checkpoint Write the definitions first and check every algebra product and idempotent. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 2: solution strategy / checkpoint Use the explicitly displayed equations and distinguish identities from existence theorems. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 3: solution strategy / checkpoint Reduce to a finite-dimensional linear-algebra calculation or a chain-level coefficient check. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 4: solution strategy / checkpoint Keep the geometric representative in minimal position; for parallel curves use the intertwiner complex. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 5: solution strategy / checkpoint Cite the exact source lemma for the nontrivial geometric step; do not assert it follows from a determinant. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 6: solution strategy / checkpoint State the category and hypothesis restrictions and give a counterexample to the stronger assertion. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

Oral examination and model answers

[!question] Question 1: What is the central result of this lecture? **Model answer:** A parametrized torus is encoded by an oriented pointed matched circle and two idempotents ι_0, ι_1.

[!question] Question 2: Which calculation here is proved completely? **Model answer:** **Derived, complete algebra check.

[!question] Question 3: What is the most important convention pitfall? **Model answer:** The algebra has zero internal differential, but the induced type D differential need not be zero.

[!question] Question 4: Why is a determinant insufficient in general? **Model answer:** A determinant counts minimal intersections of straight slope representatives, not the full Floer differential or the monodromy of parallel local systems.

[!question] Question 5: Which argument needs a source diagram? **Model answer:** The train-track, crossover-arrow or immersed-bigon equivalence step identified explicitly as a source-dependent proof sketch.

[!question] Question 6: How does the companion paper enter? **Model answer:** It supplies gradings, elliptic symmetry and manifold-specific restrictions beyond the abstract ungraded classification; the exact use depends on the assigned section.

Time plan and final verification

Segment	Minutes	Goal
Source close reading	75	Check exact statements and diagrams
Definitions and conventions	35	Restate hypotheses without notes
Proof reconstruction	60	Separate full proofs and sketches
Worked calculations	40	Check algebra independently
Six written problems	50	Produce complete or qualified answers
Oral defense	25	Six model questions

Total: 285 minutes = 4 h 45 min.

Source references: HRW23 (especially HRW23 §2.1–2.3, especially the algebra conventions and train-track definition; HRW22 §2 for grading context.); HRW22. Supporting LOT memoir is background only.