HRW Day 04 — Train Tracks and the Chain-Level Pairing

2026-10-26

Day 04 — Train Tracks and the Chain-Level Pairing

Course date: 2026-10-26. **Language:** English. **Target:** graduate qualifying examination. **Study allocation:** 4 h 45 min.

PDF attachment: [[HRW_Day04.pdf]]. **Source:** [[HRW_Day04_Source.tex]] (same folder). These are newly authored replacement notes for previously unavailable Day 04 files, not recovered copies of the original lectures.

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Reading assignment and theorem scope

Main paper: HRW23 §§2.3–2.6, Theorem 2.2 and Figures 10–12; supporting LOT pairing conventions. Consult HRW23 and HRW22. These notes follow the **source's algebraic and geometric terminology**. A statement identified as a source theorem is not claimed to have been proved in full merely because an illustrative calculation is supplied. We use the 2023 main manuscript and the 2022 published companion; theorem numbers can differ in earlier preprints.

Core definitions and exact algebraic framework

> [!theorem] Source-grounded mathematical content A type D graph can be represented by a train track in the punctured torus. A type A realization is related to the type D realization by reflection and chord-label convention changes (notably exchanging labels 1 and 3). The geometric chain group is generated by intersections of suitable train tracks, and a puncture-free immersed bigon contributes to its differential. HRW23 Theorem 2.2 compares a **reduced** train track on one side and a **special bounded** train track on the other. The latter boundedness is needed for finiteness of the box tensor differential. A chain generator corresponds to an idempotent-compatible tensor generator $x \otimes y$. An immersed bigon corresponds to a word $\rho_{I_1}, \dots, \rho_{I_k}$ in the iterated type D map and the compatible A_∞ operation $m_{k+1}(x, \rho_{I_1}, \dots, \rho_{I_k})$.

Context and motivation. Bordered Floer theory expresses gluing through a type A/type D pairing, while the immersed-curve reformulation turns the pairing into a geometric intersec-

tion problem on a once-punctured torus. The technical challenge is preserving the information of the coefficient algebra, local systems, and admissibility under graphical changes. A concrete computation must therefore be separated from a theorem asserting equivalence of whole modules.

Categories. The geometric model is a punctured **surface** with immersed curves and finite-dimensional local systems. It does not assert that every algebraically admissible curve system comes from a smooth three-manifold. All results here are over $\mathbb{F}_2$.

The main proof or proof checkpoint

> [!proof] Mathematical argument and its status **Source-dependent proof sketch, HRW23 Theorem 2.2.** Starting at an intersection $x \otimes y$, follow one boundary arc of a candidate immersed bigon along the type D train track, recording the chord labels in their order. Follow the other arc on the type A track; the same boundary word appears in the corresponding higher product. Their matching endpoints produce exactly the coefficient of the box tensor differential. Conversely, the local chord-word conditions force the boundary of a contributing immersed bigon. The proof requires an exhaustive local verification of small unit-labelled bigons and the type A/type D reflection convention (Figures 10–12); those diagrammatic cases are **not** proved by this sketch.

What to defend orally. Identify the exact algebraic equation, explain why its hypotheses hold, and distinguish the complete calculation above from the source-dependent geometric normal-form or bigon argument. The latter requires the relevant diagrams of HRW23; the text here does not silently replace those diagrams with a heuristic.

Worked computation and interpretation

For primitive straight curves $u = (1, 0)$ and $v = (3, 7)$ with trivial rank-one systems, $i(u, v) = |\det(u, v)| = 7$. Choose geodesic representatives with no immersed bigons. The differential vanishes, so $\dim HF(u, v) = 7$. This is a geometric model for the rank of an appropriate lens-space gluing, but the theorem's full chain-level content is the **differential correspondence**, not merely this determinant count.

Second check (algebraic invariant). An invertible matrix A and a basis change P define isomorphic local systems precisely when the monodromies are conjugate, $A' = P^{-1}AP$. Similarity preserves characteristic and minimal polynomials, but agreement of those two polynomials alone need not imply similarity. This observation becomes decisive in the classification proof.

Geometric check. For primitive nonparallel slope vectors $u = (a, b)$ and $v = (c, d)$, $i(u, v) = |ad - bc|$. This formula counts geometric intersections of straight representatives; when the curves are parallel, or when local-system monodromy is nontrivial, the Floer homology cannot be read from the determinant alone.

Source-specific pitfalls

> [!warning] Convention and hypothesis audit An intersection of the raw geometric pictures is not always a legitimate tensor generator unless the idempotents match. The geometric type A picture differs from the type D picture. Never ignore the special-bounded hypothesis when writing $\boxtimes$. A punctured bigon does not contribute to the hat differential.

For any argument involving closed-manifold Floer homology, the gluing homeomorphism reverses boundary orientation. For the type A/type D pairing, the elliptic-involution correction and reflection convention must be tracked; HRW22's elliptic symmetry permits simplification for **actual manifold invariants**, not automatically for every abstract curve collection.

Written qualifying examination

Instructions. Attempt these six questions without notes; write complete algebraic proofs and explicitly label any diagrammatic proof sketch.

[!exercise] Problem 1 State the hypotheses of Theorem 2.2.

[!exercise] Problem 2 Explain the intersection/tensor-generator correspondence.

[!exercise] Problem 3 Describe how an immersed bigon produces an A_∞ operation.

[!exercise] Problem 4 Compute $|\det((1,0),(3,7))|$.

[!exercise] Problem 5 Explain why the straight-curve differential vanishes.

[!exercise] Problem 6 Identify the reflection and chord-label convention pitfall.

Solutions and hints

[!proof] Problem 1: solution strategy / checkpoint Write the definitions first and check every algebra product and idempotent. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 2: solution strategy / checkpoint Use the explicitly displayed equations and distinguish identities from existence theorems. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 3: solution strategy / checkpoint Reduce to a finite-dimensional linear-algebra calculation or a chain-level coefficient check. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 4: solution strategy / checkpoint Keep the geometric representative in minimal position; for parallel curves use the intertwiner complex. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 5: solution strategy / checkpoint Cite the exact source lemma for the nontrivial geometric step; do not assert it follows from a determinant. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 6: solution strategy / checkpoint State the category and hypothesis restrictions and give a counterexample to the stronger assertion. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

Oral examination and model answers

[!question] Question 1: What is the central result of this lecture? **Model answer:** A type D graph can be represented by a train track in the punctured torus.

[!question] Question 2: Which calculation here is proved completely? **Model answer:** **Source-dependent proof sketch, HRW23 Theorem 2.

[!question] Question 3: What is the most important convention pitfall? **Model answer:** An intersection of the raw geometric pictures is not always a legitimate tensor generator unless the idempotents match.

[!question] Question 4: Why is a determinant insufficient in general? **Model answer:** A determinant counts minimal intersections of straight slope representatives, not the full Floer differential or the monodromy of parallel local systems.

[!question] Question 5: Which argument needs a source diagram? **Model answer:** The train-track, crossover-arrow or immersed-bigon equivalence step identified explicitly as a source-dependent proof sketch.

[!question] Question 6: How does the companion paper enter? **Model answer:** It supplies gradings, elliptic symmetry and manifold-specific restrictions beyond the abstract ungraded classification; the exact use depends on the assigned section.

Time plan and final verification

Segment	Minutes	Goal
Source close reading	75	Check exact statements and diagrams
Definitions and conventions	35	Restate hypotheses without notes
Proof reconstruction	60	Separate full proofs and sketches
Worked calculations	40	Check algebra independently
Six written problems	50	Produce complete or qualified answers
Oral defense	25	Six model questions

Total: 285 minutes = 4 h 45 min.

Source references: HRW23 (especially HRW23 §§2.3–2.6, Theorem 2.2 and Figures 10–12; supporting LOT pairing conventions.); HRW22. Supporting LOT memoir is background only.