HRW Day 05 — Crossover Arrows, Depth, and the Structure Theorem

2026-10-27

Day 05 — Crossover Arrows, Depth, and the Structure Theorem

Course date: 2026-10-27. **Language:** English. **Target:** graduate qualifying examination. **Study allocation:** 4 h 45 min.

PDF attachment: [[HRW_Day05.pdf]]. **Source:** [[HRW_Day05_Source.tex]] (same folder). These are newly authored replacement notes for previously unavailable Day 05 files, not recovered copies of the original lectures.

Contents

- [[#Reading assignment and theorem scope|Reading assignment and theorem scope]]
- [[#Core definitions and exact algebraic framework|Core definitions]]
- [[#The main proof or proof checkpoint|Proof checkpoint]]
- [[#Worked computation and interpretation|Worked computation]]
- [[#Source-specific pitfalls|Pitfalls]]
- [[#Written qualifying examination|Written exam]]
- [[#Solutions and hints|Solutions]]
- [[#Oral examination and model answers|Oral exam]]
- [[#Time plan and final verification|Time plan]]

Reading assignment and theorem scope

Main paper: HRW23 §§3.3–3.7: Proposition 3.8, Proposition 3.13 and Theorem 1.5. Consult HRW23 and HRW22. These notes follow the **source's algebraic and geometric terminology**. A statement identified as a source theorem is not claimed to have been proved in full merely because an illustrative calculation is supplied. We use the 2023 main manuscript and the 2022 published companion; theorem numbers can differ in earlier preprints.

Core definitions and exact algebraic framework

> [!theorem] Source-grounded mathematical content Proposition 3.8 gives a normal form: a valid reduced train track is equivalent to valence-two immersed strands, together with directed crossover arrows. The corner box contains the turns, while crossover arrows lie outside and move clockwise around it. Finite-depth arrows measure pairs of strands that agree for some distance before separating. Proposition 3.13 permits depth improvement: a configuration of depth m is equivalent to one of depth at least $m+1$. The arrows that remain after the finite-depth reduction join globally parallel strands. A packet of k such strands yields a rank-k local system, with invertible monodromy computed by directed path transport. This proves the existence part of Theorem 1.5.

Context and motivation. Bordered Floer theory expresses gluing through a type A/type D pairing, while the immersed-curve reformulation turns the pairing into a geometric intersection problem on a once-punctured torus. The technical challenge is preserving the information

of the coefficient algebra, local systems, and admissibility under graphical changes. A concrete computation must therefore be separated from a theorem asserting equivalence of whole modules.

Categories. The geometric model is a punctured **surface** with immersed curves and finite-dimensional local systems. It does not assert that every algebraically admissible curve system comes from a smooth three-manifold. All results here are over $\mathbb{F}_2$.

The main proof or proof checkpoint

> [!proof] Mathematical argument and its status **Source-dependent proof sketch, not a complete arrow-slide proof.** First put the train track in Proposition 3.8 normal form by elementary changes of basis. Sort crossover arrows by their forward/backward agreement lengths. At the minimum finite depth, use the moves of HRW23 §3.6 to slide and cancel those arrows. When one arrow crosses another, the composition arrow must be included; Proposition 3.13 verifies that no new arrow of smaller depth appears. Repeat until the only surviving arrows join globally parallel strands. Choose a transverse cut, and use the directed path matrix to define the monodromy. The full termination and local-case verification depend on HRW23's figures and are not proved here.

What to defend orally. Identify the exact algebraic equation, explain why its hypotheses hold, and distinguish the complete calculation above from the source-dependent geometric normal-form or bigon argument. The latter requires the relevant diagrams of HRW23; the text here does not silently replace those diagrams with a heuristic.

Worked computation and interpretation

A double traversal of a primitive circle with trivial rank-one local system induces on the primitive curve the rank-two monodromy $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Over $\mathbb{F}_2$, $S^2 = I$ but $S \neq I$, and $m_S(t) = (t+1)^2$ whereas $m_{I_2}(t) = t + 1$. Hence two trivial rank-one parallel curves and a trivial double cover are not equivalent as decorated curves.

Second check (algebraic invariant). An invertible matrix A and a basis change P define isomorphic local systems precisely when the monodromies are conjugate, $A' = P^{-1}AP$. Similarity preserves characteristic and minimal polynomials, but agreement of those two polynomials alone need not imply similarity. This observation becomes decisive in the classification proof.

Geometric check. For primitive nonparallel slope vectors $u = (a, b)$ and $v = (c, d)$, $i(u, v) = |ad - bc|$. This formula counts geometric intersections of straight representatives; when the curves are parallel, or when local-system monodromy is nontrivial, the Floer homology cannot be read from the determinant alone.

Source-specific pitfalls

> [!warning] Convention and hypothesis audit A crossover arrow cannot be erased just because the underlying curves appear unchanged. Infinite-depth arrows can carry nontrivial monodromy. Theorem 1.5 is an **existence** theorem, not the uniqueness theorem: injectivity is proved in §4, using Floer test curves.

For any argument involving closed-manifold Floer homology, the gluing homeomorphism reverses boundary orientation. For the type A/type D pairing, the elliptic-involution correction and reflection convention must be tracked; HRW22's elliptic symmetry permits simplification for **actual manifold invariants**, not automatically for every abstract curve collection.

Written qualifying examination

Instructions. Attempt these six questions without notes; write complete algebraic proofs and explicitly label any diagrammatic proof sketch.

> [!exercise] Problem 1 State Proposition 3.8 with its geometric normal-form features.

[!exercise] Problem 2 Explain the meaning of finite versus infinite depth.

[!exercise] Problem 3 Describe why composition arrows arise during slides.

[!exercise] Problem 4 State Proposition 3.13 and its role in the induction.

[!exercise] Problem 5 Compute the induced monodromy for a double cover.

[!exercise] Problem 6 Distinguish Theorem 1.5 from Theorem 4.11.

Solutions and hints

[!proof] Problem 1: solution strategy / checkpoint Write the definitions first and check every algebra product and idempotent. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 2: solution strategy / checkpoint Use the explicitly displayed equations and distinguish identities from existence theorems. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 3: solution strategy / checkpoint Reduce to a finite-dimensional linear-algebra calculation or a chain-level coefficient check. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 4: solution strategy / checkpoint Keep the geometric representative in minimal position; for parallel curves use the intertwiner complex. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 5: solution strategy / checkpoint Cite the exact source lemma for the nontrivial geometric step; do not assert it follows from a determinant. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 6: solution strategy / checkpoint State the category and hypothesis restrictions and give a counterexample to the stronger assertion. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

Oral examination and model answers

[!question] Question 1: What is the central result of this lecture? **Model answer:** Proposition 3.

[!question] Question 2: Which calculation here is proved completely? **Model answer:** **Source-dependent proof sketch, not a complete arrow-slide proof.

[!question] Question 3: What is the most important convention pitfall? **Model answer:** A crossover arrow cannot be erased just because the underlying curves appear unchanged.

[!question] Question 4: Why is a determinant insufficient in general? **Model answer:** A determinant counts minimal intersections of straight slope representatives, not the full Floer differential or the monodromy of parallel local systems.

[!question] Question 5: Which argument needs a source diagram? **Model answer:** The train-track, crossover-arrow or immersed-bigon equivalence step identified explicitly as a source-dependent proof sketch.

[!question] Question 6: How does the companion paper enter? **Model answer:** It supplies gradings, elliptic symmetry and manifold-specific restrictions beyond the abstract ungraded classification; the exact use depends on the assigned section.

Time plan and final verification

Segment	Minutes	Goal
Source close reading	75	Check exact statements and diagrams
Definitions and conventions	35	Restate hypotheses without notes
Proof reconstruction	60	Separate full proofs and sketches
Worked calculations	40	Check algebra independently
Six written problems	50	Produce complete or qualified answers
Oral defense	25	Six model questions

Total: 285 minutes = 4 h 45 min.

Source references: HRW23 (especially HRW23 §§3.3–3.7: Proposition 3.8, Proposition 3.13 and Theorem 1.5.); HRW22. Supporting LOT memoir is background only.