HRW Day 06 — Decorated Curves, Admissibility, and Local-System Floer Theory

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Course date: 2026-10-28. **Language:** English. **Target:** graduate qualifying examination. **Study allocation:** 4 h 45 min.

PDF attachment: [[HRW_Day06.pdf]]. **Source:** [[HRW_Day06_Source.tex]] (same folder). These are newly authored replacement notes for previously unavailable Day 06 files, not recovered copies of the original lectures.

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Reading assignment and theorem scope

Main paper: HRW23 §4.1–4.2: Definitions 4.1, 4.2, 4.7, Lemma 4.8, Corollary 4.10, Theorem 4.11. Consult HRW23 and HRW22. These notes follow the **source's algebraic and geometric terminology**. A statement identified as a source theorem is not claimed to have been proved in full merely because an illustrative calculation is supplied. We use the 2023 main manuscript and the 2022 published companion; theorem numbers can differ in earlier preprints.

Core definitions and exact algebraic framework

> [!theorem] Source-grounded mathematical content A decorated component is an unobstructed immersed loop in $T^2 \setminus \{z\}$ with a finite-dimensional local system. Unobstructedness means its bi-infinite lift to the universal cover is embedded (HRW23 Definition 4.1). Commensurable loops have nonzero powers representing the same free homotopy class. Incommensurable geodesic representatives admit a no-bigon Floer model, with $\dim HF = kk'i(\gamma, \gamma')$ for local-system ranks k, k'. For commensurable curves, periodic annular domains may appear, and HRW23 Definition 4.7 imposes an admissibility condition: the relevant periodic domain must have both positive and negative multiplicities. Parallel primitive circles can have $i = 0$ but nonzero Floer homology.

Context and motivation. Bordered Floer theory expresses gluing through a type A/type D pairing, while the immersed-curve reformulation turns the pairing into a geometric intersection problem on a once-punctured torus. The technical challenge is preserving the information

of the coefficient algebra, local systems, and admissibility under graphical changes. A concrete computation must therefore be separated from a theorem asserting equivalence of whole modules.

Categories. The geometric model is a punctured **surface** with immersed curves and finite-dimensional local systems. It does not assert that every algebraically admissible curve system comes from a smooth three-manifold. All results here are over $\mathbb{F}_2$.

The main proof or proof checkpoint

> [!proof] Mathematical argument and its status **Derived, complete algebra calculation for embedded parallel curves.** Let $A \in GL(V)$ and $C \in GL(W)$. The two-term circle cochain model with coefficients $\mathrm{Hom}(W, V)$ has differential $d(X) = AX - XC$ (or an equivalent conjugated form). Its kernel is the intertwiner space $\{X : AX = XC\}$. Both cochain groups have dimension kk', and if $r = \operatorname{rank} d$, then $\dim H^0 = kk' - r$ and $\dim H^1 = kk' - r$. Thus $\dim H^* = 2 \dim \ker d$. For trivial rank-one systems $A = C = 1$ over $\mathbb{F}_2$, $d = 0$ and the Floer dimension is 2. This proves the algebraic rank formula; its identification with the admissible immersed Floer model is the cited HRW geometric input.

What to defend orally. Identify the exact algebraic equation, explain why its hypotheses hold, and distinguish the complete calculation above from the source-dependent geometric normal-form or bigon argument. The latter requires the relevant diagrams of HRW23; the text here does not silently replace those diagrams with a heuristic.

Worked computation and interpretation

Let $A = I_2$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Pair each with a rank-one trivial parallel test. The first has $\dim \ker(I + A) = 2$, giving rank 4. The second has $I + B = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ invertible, giving rank 0. Hence equal geometric support and rank do not imply equal decorated-curve objects.

Second check (algebraic invariant). An invertible matrix A and a basis change P define isomorphic local systems precisely when the monodromies are conjugate, $A' = P^{-1}AP$. Similarity preserves characteristic and minimal polynomials, but agreement of those two polynomials alone need not imply similarity. This observation becomes decisive in the classification proof.

Geometric check. For primitive nonparallel slope vectors $u = (a, b)$ and $v = (c, d)$, $i(u, v) = |ad - bc|$. This formula counts geometric intersections of straight representatives; when the curves are parallel, or when local-system monodromy is nontrivial, the Floer homology cannot be read from the determinant alone.

Source-specific pitfalls

> [!warning] Convention and hypothesis audit Do not confuse homology slopes with free homotopy classes in the punctured torus: $\pi_1(T^2 \setminus \{z\}) \cong F_2$ is nonabelian. Likewise, a nonprimitive multiple cover is not just a direct sum of primitive copies. The parallel Floer formula requires the correct orientation/dualization convention for monodromy.

For any argument involving closed-manifold Floer homology, the gluing homeomorphism reverses boundary orientation. For the type A/type D pairing, the elliptic-involution correction and reflection convention must be tracked; HRW22's elliptic symmetry permits simplification for **actual manifold invariants**, not automatically for every abstract curve collection.

Written qualifying examination

Instructions. Attempt these six questions without notes; write complete algebraic proofs and explicitly label any diagrammatic proof sketch.

> [!exercise] Problem 1 State unobstructedness and commensurability.

> [!exercise] Problem 2 Explain the annular admissibility issue.

[!exercise] Problem 3 Compute Floer rank for nonparallel primitive slopes $(2, 3)$ and $(1, 4)$ with ranks two and three.

[!exercise] Problem 4 Derive the two-term circle cochain formula.

[!exercise] Problem 5 Distinguish I_2 and the irreducible rank-two monodromy using a parallel test.

[!exercise] Problem 6 Explain why equal homology classes need not imply equal free homotopy classes.

Solutions and hints

[!proof] Problem 1: solution strategy / checkpoint Write the definitions first and check every algebra product and idempotent. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 2: solution strategy / checkpoint Use the explicitly displayed equations and distinguish identities from existence theorems. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 3: solution strategy / checkpoint Reduce to a finite-dimensional linear-algebra calculation or a chain-level coefficient check. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 4: solution strategy / checkpoint Keep the geometric representative in minimal position; for parallel curves use the intertwiner complex. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 5: solution strategy / checkpoint Cite the exact source lemma for the nontrivial geometric step; do not assert it follows from a determinant. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 6: solution strategy / checkpoint State the category and hypothesis restrictions and give a counterexample to the stronger assertion. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

Oral examination and model answers

[!question] Question 1: What is the central result of this lecture? **Model answer:** A decorated component is an unobstructed immersed loop in $T^2 \setminus \{z\}$ with a finite-dimensional local system.

[!question] Question 2: Which calculation here is proved completely? **Model answer:** **Derived, complete algebra calculation for embedded parallel curves.

[!question] Question 3: What is the most important convention pitfall? **Model answer:** Do not confuse homology slopes with free homotopy classes in the punctured torus: $\pi_1(T^2 \setminus \{z\}) \cong F_2$ is nonabelian.

[!question] Question 4: Why is a determinant insufficient in general? **Model answer:** A determinant counts minimal intersections of straight slope representatives, not the full Floer differential or the monodromy of parallel local systems.

[!question] Question 5: Which argument needs a source diagram? **Model answer:** The train-track, crossover-arrow or immersed-bigon equivalence step identified explicitly as a source-dependent proof sketch.

[!question] Question 6: How does the companion paper enter? **Model answer:** It supplies gradings, elliptic symmetry and manifold-specific restrictions beyond the abstract ungraded classification; the exact use depends on the assigned section.

Time plan and final verification

Segment	Minutes	Goal
Source close reading	75	Check exact statements and diagrams
Definitions and conventions	35	Restate hypotheses without notes
Proof reconstruction	60	Separate full proofs and sketches
Worked calculations	40	Check algebra independently
Six written problems	50	Produce complete or qualified answers
Oral defense	25	Six model questions

Total: 285 minutes = 4 h 45 min.

Source references: HRW23 (especially HRW23 §4.1–4.2: Definitions 4.1, 4.2, 4.7, Lemma 4.8, Corollary 4.10, Theorem 4.11.); HRW22. Supporting LOT memoir is background only.