HRW Day 07 — Injectivity, Test Curves, and Geometric Pairing

2026-10-29

Day 07 — Injectivity, Test Curves, and Geometric Pairing

Course date: 2026-10-29. **Language:** English. **Target:** graduate qualifying examination. **Study allocation:** 4 h 45 min.

PDF attachment: [[HRW_Day07.pdf]]. **Source:** [[HRW_Day07_Source.tex]] (same folder). These are newly authored replacement notes for previously unavailable Day 07 files, not recovered copies of the original lectures.

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Reading assignment and theorem scope

Main paper: HRW23 §§4.2–5: Theorem 4.11, Lemma 4.14, Proposition 4.15, Proposition 4.16 and parametrization invariance; HRW22 Theorem 7. Consult HRW23 and HRW22. These notes follow the **source's algebraic and geometric terminology**. A statement identified as a source theorem is not claimed to have been proved in full merely because an illustrative calculation is supplied. We use the 2023 main manuscript and the 2022 published companion; theorem numbers can differ in earlier preprints.

Core definitions and exact algebraic framework

> [!theorem] Source-grounded mathematical content Theorem 4.11 is the pairing-compatible classification of extendable type D modules by unobstructed decorated multicurves. Surjectivity follows from the structure theorem. Injectivity has two stages: open weakly-valid test tracks detect weighted cyclic subword counts, hence the free homotopy classes of components (Lemma 4.14); parallel decorated test curves detect the similarity classes of monodromy matrices (Proposition 4.15). The latter uses the rational canonical form over $R = \mathbb{F}_2[t, t^{-1}]$. Parametrization invariance for manifold invariants is separate from the abstract classification; the elliptic-involution simplification is supplied by HRW22 Theorem 7 for actual manifold curves.

Context and motivation. Bordered Floer theory expresses gluing through a type A/type D pairing, while the immersed-curve reformulation turns the pairing into a geometric intersection problem on a once-punctured torus. The technical challenge is preserving the information

of the coefficient algebra, local systems, and admissibility under graphical changes. A concrete computation must therefore be separated from a theorem asserting equivalence of whole modules.

Categories. The geometric model is a punctured **surface** with immersed curves and finite-dimensional local systems. It does not assert that every algebraically admissible curve system comes from a smooth three-manifold. All results here are over $\mathbb{F}_2$.

The main proof or proof checkpoint

> [!proof] Mathematical argument and its status **Derived, complete local-system separation argument.** Regard $A \in GL_k(\mathbb{F})$ as a torsion module V_A over $R = \mathbb{F}[t, t^{-1}]$, decomposed into primary blocks $R/(p^{a_j})$. For an irreducible p of degree d with $p(0) \neq 0$, a map $R/(p^\ell) \to R/(p^a)$ is determined by an image x killed by p^ℓ; its dimension is $d \min(\ell, a)$. Choose a test matrix $B = (C_{p,\ell}^{-1})^T$, with $C_{p,\ell}$ the companion matrix of p^ℓ. Then $\dim \ker(I + A \otimes B) = d \sum_j \min(\ell, a_j)$. Taking successive differences in ℓ counts blocks of length at least ℓ, hence recovers every elementary divisor. Nonsimilar matrices are distinguished by some test. This proves the algebraic heart of Proposition 4.15; the translation to Floer test curves uses HRW23's pairing theorem.

What to defend orally. Identify the exact algebraic equation, explain why its hypotheses hold, and distinguish the complete calculation above from the source-dependent geometric normal-form or bigon argument. The latter requires the relevant diagrams of HRW23; the text here does not silently replace those diagrams with a heuristic.

Worked computation and interpretation

Let $A_1 = J_4 \oplus J_2 \oplus J_2$ and $A_2 = J_4 \oplus J_3 \oplus J_1$ with unipotent Jordan blocks over $\mathbb{F}_2$. Both have characteristic polynomial $(t+1)^8$ and minimal polynomial $(t+1)^4$. The $p = t + 1$, $\ell = 2$ test gives kernel dimensions 6 and 5 on a sixteen-dimensional tensor space; thus ranks 10 and 11. One cannot distinguish these matrices merely by their characteristic or minimal polynomial.

Second check (algebraic invariant). An invertible matrix A and a basis change P define isomorphic local systems precisely when the monodromies are conjugate, $A' = P^{-1}AP$. Similarity preserves characteristic and minimal polynomials, but agreement of those two polynomials alone need not imply similarity. This observation becomes decisive in the classification proof.

Geometric check. For primitive nonparallel slope vectors $u = (a, b)$ and $v = (c, d)$, $i(u, v) = |ad - bc|$. This formula counts geometric intersections of straight representatives; when the curves are parallel, or when local-system monodromy is nontrivial, the Floer homology cannot be read from the determinant alone.

Source-specific pitfalls

> [!warning] Convention and hypothesis audit The open word-test tracks of Lemma 4.14 are not necessarily extendable manifold objects. That is acceptable: they are algebraic probes. Also, Proposition 4.16's uniqueness of an extension is a **consequence** of classification, not an assumption used to prove injectivity.

For any argument involving closed-manifold Floer homology, the gluing homeomorphism reverses boundary orientation. For the type A/type D pairing, the elliptic-involution correction and reflection convention must be tracked; HRW22's elliptic symmetry permits simplification for **actual manifold invariants**, not automatically for every abstract curve collection.

Written qualifying examination

Instructions. Attempt these six questions without notes; write complete algebraic proofs and explicitly label any diagrammatic proof sketch.

> [!exercise] Problem 1 State Theorem 4.11 as a bijection of equivalence classes.

[!exercise] Problem 2 Explain why cyclic-word tests are needed in addition to slope determinants.

[!exercise] Problem 3 Prove the Hom-space formula for $R/(p^{\ell})$ and $R/(p^{a})$.

[!exercise] Problem 4 Compute the J_2 test nullities for the two eight-dimensional matrices.

[!exercise] Problem 5 Distinguish Theorems 1.5, 2.2 and 4.11.

[!exercise] Problem 6 Explain the manifold-specific elliptic-involution correction.

Solutions and hints

[!proof] Problem 1: solution strategy / checkpoint Write the definitions first and check every algebra product and idempotent. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 2: solution strategy / checkpoint Use the explicitly displayed equations and distinguish identities from existence theorems. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 3: solution strategy / checkpoint Reduce to a finite-dimensional linear-algebra calculation or a chain-level coefficient check. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 4: solution strategy / checkpoint Keep the geometric representative in minimal position; for parallel curves use the intertwiner complex. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 5: solution strategy / checkpoint Cite the exact source lemma for the nontrivial geometric step; do not assert it follows from a determinant. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 6: solution strategy / checkpoint State the category and hypothesis restrictions and give a counterexample to the stronger assertion. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

Oral examination and model answers

[!question] Question 1: What is the central result of this lecture? **Model answer:** Theorem 4.

[!question] Question 2: Which calculation here is proved completely? **Model answer:** **Derived, complete local-system separation argument.

[!question] Question 3: What is the most important convention pitfall? **Model answer:** The open word-test tracks of Lemma 4.

[!question] Question 4: Why is a determinant insufficient in general? **Model answer:** A determinant counts minimal intersections of straight slope representatives, not the full Floer differential or the monodromy of parallel local systems.

[!question] Question 5: Which argument needs a source diagram? **Model answer:** The train-track, crossover-arrow or immersed-bigon equivalence step identified explicitly as a source-dependent proof sketch.

[!question] Question 6: How does the companion paper enter? **Model answer:** It supplies gradings, elliptic symmetry and manifold-specific restrictions beyond the abstract ungraded classification; the exact use depends on the assigned section.

Time plan and final verification

Segment	Minutes	Goal
Source close reading	75	Check exact statements and diagrams
Definitions and conventions	35	Restate hypotheses without notes
Proof reconstruction	60	Separate full proofs and sketches
Worked calculations	40	Check algebra independently
Six written problems	50	Produce complete or qualified answers
Oral defense	25	Six model questions

Total: 285 minutes = 4 h 45 min.

Source references: HRW23 (especially HRW23 §§4.2–5: Theorem 4.11, Lemma 4.14, Proposition 4.15, Proposition 4.16 and parametrization invariance; HRW22 Theorem 7.); HRW22. Supporting LOT memoir is background only.