Day 09 — Dehn Twists, the Elliptic Involution, and Spin-c Conjugation

Course date: 31 October 2026. **Duration:** 4 h 50 min. **Scope:** HRW23 §5 and HRW22 §3 (both primary). The previous day established refined gradings; today we study how the curve invariant behaves under changes of parametrization, orientation reversal, and conjugation, and why these properties matter for gluing.

> **Warning.** Source and notation policy We write $\mathscr{C}(M) = \widehat{HF}(M)$ for the **decorated curve set**, while $\widehat{HF}(Y)$ denotes the Heegaard Floer **vector space** of a closed manifold. The former is not an ordinary group. Theorem numbers for the *published* HRW22 article are used: Theorem 38, Corollaries 39 and 42, Proposition 40. In the 2018 arXiv v1 these are numbered Theorem 37, Corollary 38, and Proposition 39, respectively. A theorem of the papers is marked **[SOURCE]**; an auxiliary calculation supplied here is marked **[DERIVED]**. A proof outline that does not reproduce the figure-dependent cases is expressly marked **[PROOF SKETCH]**.

Study map

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Connections: `Torus Algebra`, `Type D Structures`, `Mapping Class Group of a Torus`, `Crossover Arrows`, `Local Systems`, `Spin-c Conjugation`, `Genus One Mutation`, `Dehn Filling`. **Companion PDF:** HRW_Day09.pdf.

1. The global logical picture

Let M be a compact oriented 3-manifold with parametrized torus boundary and a distinguished puncture z. Write

$$T_M = H_1(\partial M;\mathbb{R})/H_1(\partial M;\mathbb{Z}) \smallsetminus \{0\}.$$

The identification of this abstract punctured torus with a standard square depends on a basis (α,β) of the boundary lattice. The underlying curve invariant must not depend on that auxiliary choice. The previous lectures established the equivalence between extendable torus-algebra type D structures and unobstructed immersed multicurves with local systems (HRW23, Theorem 4.11). Two further tasks remain: remove dependence on the boundary basis, and align the geometric pairing with actual gluing.

> **Theorem.** [SOURCE] HRW23, Theorems 1.1 and 1.2 The decorated curve set $\mathscr{C}(M)$ is independent of a chosen boundary parametrization, up to regular homotopy and local-system isomorphism. If $Y = M_0 \cup_h M_1$ and $h : \partial M_1 \to \partial M_0$ reverses orientation and identifies punctures, then the 2023 formulation is
>
> $$\widehat{HF}(Y) \cong HF\Big(\mathscr{C}(M_0), \overline{h}(\mathscr{C}(M_1))\Big),$$
>
> where $\overline{h}$ is the gluing map composed with the elliptic involution. The 2022 companion paper proves that the elliptic involution preserves $\mathscr{C}(M)$, so $\overline{h}$ can be replaced by h.

Here the intersection Floer complex includes local-system transport, and admissible transverse representatives are used. The pairing formula is not merely the rank of a raw intersection set whenever parallel components or nontrivial local systems occur. The grading refinement from Day 8 has additional hypotheses and decorations; this lecture treats the ungraded symmetry statement unless stated otherwise.

The central diagram of arguments is

$$\begin{array}{ccc} \widehat{CFD}(M,\alpha,\beta) & \xrightarrow{\text{HRW23, §4}} & \mathscr{C}_{\alpha,\beta}(M) \\ \downarrow\text{Dehn-twist DA bimodule} & & \downarrow\text{geometric twist} \\ \widehat{CFD}(M,\alpha',\beta') & \xrightarrow{\text{HRW23, §4}} & \mathscr{C}_{\alpha',\beta'}(M). \end{array}$$

The other half of the lecture compares the algebraic conjugation functor $F_{\mathbf{A}}$ with the geometric elliptic functor $F_{\mathbf{E}}$. These functors are **not equivalent on the whole abstract category**; they agree on the subcategory realized by actual bordered invariants of 3-manifolds. The peripheral-circle obstruction is responsible for this restriction.

2. The mapping class group and elementary Dehn twists

Choose a positively oriented ordered basis (a, b) of $H_1(T^2;\mathbb{Z})$ and identify a primitive class $pa+qb$ with the column vector $(p,q)^T$. Use the standard intersection convention $a \cdot b = +1$. The following matrices define a consistent set of **algebraic** twist conventions:

$$T_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad T_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{2.1}$$

They correspond to $b \mapsto b + a$ and $a \mapsto a - b$, respectively. When comparing with HRW's α, β drawings, check the authors' signed intersection $\beta \cdot \alpha = 1$ and whether a twist acts on the parametrization or on the drawn curve: that may reverse which twist is called positive.

Proposition. [DERIVED] Elementary twists generate $SL(2,\mathbb{Z})$ The matrices T_a, T_b generate $SL(2,\mathbb{Z})$. Moreover,

$$(T_a T_b)^3 = -I. \tag{2.2}$$

Proof. Proof Multiplication gives $T_a T_b = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, whose cube is $-I$. To prove generation, left-multiplying a primitive column $(p,q)^T$ by integral powers of T_a or T_b implements the subtractive Euclidean algorithm $(p,q) \mapsto (p+nq, q)$ or $(p,q) \mapsto (p, q-np)$. Reduce the first column of a matrix in $SL(2,\mathbb{Z})$ to $(1,0)^T$ (up to a sign; $-I$ is already in the subgroup). The remaining matrix is then $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right) = T_a^k$. Reverse the reductions. □

The quotient $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm I\}$ **forgets** the central involution. We cannot pass to it when working with a punctured torus whose marked point is fixed. The map $\epsilon([v]) = [-v]$ has order two, sends each homology class v to $-v$, and fixes the origin puncture. It is an orientation-preserving 180° rotation, not a reflection.

2.1. Determinants, slopes, and intersections

For primitive classes $v = (p,q)$ and $w = (r,s)$, the minimal geometric intersection of embedded representatives is

$$i(v,w) = |\det(v,w)| = |ps - qr|. \tag{2.3}$$

For $A \in GL(2,\mathbb{Z})$ with $|\det A| = 1$,

$$|\det(Av, Aw)| = |\det A|\,|\det(v,w)| = i(v,w).$$

In particular, $i(v,-w) = i(v,w)$, but this **does not prove** elliptic invariance of an arbitrary immersed multicurve with local system. A single determinant only detects homology classes of embedded essential circles; it sees neither self-intersections nor monodromy. The geometric theorem is genuinely stronger.

3. Parametrization independence: HRW23, Proposition 5.1

Let $f_{\alpha,\beta}(N)$ denote the decorated curves constructed from the torus-algebra type D structure N, drawn in the actual boundary using the parametrization (α,β). HRW23, §5, first establishes independence under the two elementary changes of basis.

Theorem. [SOURCE] HRW23, Proposition 5.1 There are equivalences of decorated curves

$$f_{\alpha,\beta}\Big(\widehat{CFD}(M,\alpha,\beta)\Big) \simeq f_{\alpha,\beta\pm\alpha}\Big(\widehat{CFD}(M,\alpha,\beta\pm\alpha)\Big),$$
$$f_{\alpha,\beta}\Big(\widehat{CFD}(M,\alpha,\beta)\Big) \simeq f_{\alpha\pm\beta,\beta}\Big(\widehat{CFD}(M,\alpha\pm\beta,\beta)\Big).$$

The equivalences are regular homotopies of curves together with isomorphisms of their local systems.

3.1. What is really being compared?

There are two operations that must be distinguished. **Active** twisting applies a diffeomorphism to the drawn curve in a fixed coordinate square. **Passive** twisting changes which boundary basis the square represents. If an active twist changes the curve by τ^{-1} while a passive change modifies the marking by τ, their effects cancel in the actual surface. Writing only one of these operations leads to the incorrect claim that the curve *has moved* as a manifold invariant.

Let $\widehat{T}_\alpha$ be the bordered type DA bimodule for the Dehn twist used in HRW23, Figure 42. The pairing with this bimodule changes the algebraic boundary marking. In the chosen square, the same bimodule acts on a curve-like train track by a geometric twist of the opposite sign. This is verified by following the arrows labeled $\rho_2, \rho_{23}, \rho_{12}, \rho_{123}$ and cancelling the short unit-labeled arrows arising from the move.

Proof. [PROOF SKETCH] Structure of HRW23's proof Start with one decorated curve γ and expand its local system as a packet of parallel strands with crossover arrows. Slide the crossover-arrow box away from the short region where the twist bimodule acts. After tensoring with $\widehat{T}_\alpha$, every old ι_0 switch gains a new ι_1 switch connected by a ρ_2 arrow; suitable edges ending at the old switch change $\rho_2 \mapsto \rho_{23}$ and $\rho_{12} \mapsto \rho_{123}$. The resulting train track differs from the geometrically twisted train track only by local bigons and unit arrows. Their cancellations (HRW23, Figures 43–44) yield the same reduced curve with the transported local system. The twist of the marking cancels the twist of the curve. The proof for the other basis generator is analogous; the paper explicitly leaves one of these two demonstrations to the reader.

3.2. Consequence

Because elementary shears generate $SL(2,\mathbb{Z})$, Proposition 5.1 proves independence from an arbitrary oriented basis. Combined with Theorem 4.11 (classification of type D structures) and Theorem 1.4 (extendability for manifold invariants), this completes the proof of HRW23, Theorem 1.1. One does **not** need a separate invariance proof for every mapping-class element.

Proof checkpoint. Explain why the existence of a geometric twist action on slope classes is insufficient here: we must compare the *entire type DA action*, including higher A_∞ operations, unit cancellations and local-system transport.

4. The elliptic involution and the gluing map

Let $Y = M_0 \cup_h M_1$, with h orientation reversing and $h(z_1) = z_0$. The 2023 proof of Theorem 1.2 chooses bordered parametrizations in which the type A realization of one input is reflected across the anti-diagonal before its intersections with the type D realization are counted. In the actual boundary coordinates, this reflection represents the gluing map composed with ϵ.

Theorem. [SOURCE] HRW23, Theorem 1.2 and Remark 5.2 With $\overline{h} = \epsilon \circ h$ (equivalently the corresponding composition after transporting ϵ between tori), the original geometric pairing identifies

$$\widehat{HF}(Y) \cong HF\Big(\mathscr{C}(M_0), \overline{h}(\mathscr{C}(M_1))\Big). \tag{4.1}$$

The source of ϵ is **not** a geometric mutation performed on Y. It is a convention built into converting a type D structure on one piece to the type A structure required by the LOT box tensor product. This is why the 2023 paper can prove the pairing before proving elliptic invariance.

Later, HRW22 establishes $\epsilon\mathscr{C}(M) \simeq \mathscr{C}(M)$ for actual three-manifold invariants, so (4.1) becomes

$$\boxed{\widehat{HF}(Y) \cong HF\Big(\mathscr{C}(M_0), h(\mathscr{C}(M_1))\Big).} \tag{4.2}$$

Warning. Do not remove ϵ prematurely The elliptic symmetry is **not** a formal property of every abstract extendable type D structure or every arbitrary unobstructed curve with local system. HRW22 requires a special restriction to curves realized by actual 3-manifolds, proved using extra minus-type bordered information. Without that restriction, replacing $\overline{h}$ with h is unjustified.

5. Orientation reversal is not conjugation

There are two distinct operations that can be confused in diagrams.

Orientation reversal: replacing M by $-M$ reverses the orientation of its boundary. HRW22, §3.1 explains that the abstract decorated curves in $\partial M \setminus z$ stay the same, but a drawing in the *standard oriented square* changes by an orientation-reversing reflection. With their convention that the horizontal direction is the rational longitude, the reflection reverses the other coordinate. A right-handed and left-handed trefoil complement provide an example (HRW22, Figure 25 in the 2018 preprint; figure numbering in the journal should be followed there).

Spin-c conjugation: $c : \mathrm{Spin}^c(M) \to \mathrm{Spin}^c(M)$ conjugates a spin-c structure without reversing the orientation of M. The associated geometric operation is the orientation-preserving elliptic involution, which acts by $-I$ on boundary first homology. The statement relates the summand $\mathscr{C}(M, \mathfrak{s})$ to $\mathscr{C}(M, c(\mathfrak{s}))$; it does not say that each *labeled* summand is individually fixed.

An orientation-reversing reflection R has $\det R = -1$. The elliptic involution $\epsilon = -I$ has $\det \epsilon = +1$. Their effects on oriented intersection signs, and on the matching of spin-c labels, differ. In particular, a picture symmetric under a vertical reflection does not automatically establish symmetry under a half-turn.

6. The three bimodules A, E, and I

HRW22, §3.2 introduces three type DA bimodules, used as endofunctors of the relevant category of bordered modules:

- **A**: the bimodule implementing **spin-c conjugation**, obtained from the torus algebra considered as a bimodule together with the identity DD bimodule. This is the algebraic action arising from the LOT conjugation symmetry.
- **E**: the mapping-cylinder bimodule for the **elliptic involution** of the punctured torus.
- **I**: the **half-identity** bimodule, a technically useful involutive correction.

The letters are merely names, and **I** is not the identity bimodule.

Theorem. [SOURCE] HRW22, Theorem 38 and Proposition 40 (published numbering) For every 3-manifold M with torus boundary and every $\mathfrak{s} \in \mathrm{Spin}^c(M)$,

$$\widehat{CFD}(M, c(\mathfrak{s})) \simeq \mathbf{E} \boxtimes \widehat{CFD}(M, \mathfrak{s}). \tag{6.1}$$

The proof compares the three bimodules via relations represented by the commutative diagram of Proposition 40; in particular one can model **E** by composing **A** with **I**, and $\mathbf{I} \boxtimes \mathbf{I}$ acts as the identity bimodule.

For an appropriate consistent order of DA tensor products, we have the schematic relations

$$F_{\mathbf{E}} \simeq F_{\mathbf{A}} \circ F_{\mathbf{I}}, \qquad F_{\mathbf{I}}^2 \simeq \mathrm{Id}. \tag{6.2}$$

The precise left/right and opposite-algebra placements in the original statements matter. It is safer to remember (6.2) as the **functorial architecture** and consult HRW22, Figures 26–28, before performing an explicit DA-box tensor calculation.

6.1. The algebraic obstruction to simply declaring A=E

HRW22, Figure 27 shows that **A** and **E** differ by a specific higher operation (marked as a dashed edge). They are **not isomorphic as unrestricted type DA bimodules**. The authors demonstrate their inequivalence by applying them to an abstract type D structure represented by a small loop $\mathscr{P}$ enclosing the puncture (Figures 31–32).

This distinction makes the argument substantive. Proving $F_{\mathbf{E}}(N) \simeq F_{\mathbf{A}}(N)$ **for bordered invariants of manifolds** requires showing that their difference becomes invisible on that subcategory. HRW22 does this by proving $F_{\mathbf{I}}(N) \simeq N$ for precisely those objects.

7. Coils, finite-depth arrows, and the exceptional peripheral circle

A **coil** in the graph of a curve is the local succession of labels $3, 2, 1$ appearing in HRW22, §3.2. The extra operation in the half-identity bimodule creates an additional ρ_{123} arrow at such a coil. This arrow is represented geometrically as a crossover arrow.

For a curve not freely homotopic to the small peripheral loop $\mathscr{P}$ enclosing the puncture, the new arrow has **finite depth**. The first such arrow has forward weight $\check{w} = 1$; consecutive coils produce examples with $\check{w} = 1, 5, 9, \ldots$ (HRW22, Figure 34). As in Day 5, finite-depth arrows with a favorable clockwise sign are eliminated by the crossover-arrow reduction algorithm. If a nontrivial local system is present, first expand it away from the coils, so its arrows do not interfere with the local cancellation. Any new arrows formed by composition are still removable in the same depth filtration.

> **Proposition.** [SOURCE, proved as part of HRW22, Theorem 38] For an immersed component γ not homotopic to the puncture-linking circle $\mathscr{P}$, with an arbitrary local system, the half-identity functor acts trivially on its isomorphism class:
>
> $$F_{\mathbf{I}}(\gamma, \Phi) \simeq (\gamma, \Phi). \tag{7.1}$$
>
> **Proof.** [PROOF SKETCH: figure-dependent arrow cancellation] Move all local-system crossover arrows into a packet away from the coils. Tensoring with **I** leaves non-coil segments unchanged but adds a ρ_{123} arrow at each coil. Except on a curve parallel to the puncture, each newly inserted arrow joins strands with different eventual itineraries, hence has finite depth. Apply HRW23's depth-increasing moves in order of increasing depth; the $\check{w} > 0$ orientation permits the corresponding removal. Composites with the previously isolated local-system arrows have the same favorable weight and are removed as well. This is HRW22, proof of Theorem 38, referring to HRW23, §3.7. The detailed tile and arrow calculations are in HRW22, Figures 33–34.

For $\mathscr{P}$ the argument fails: the newly created arrows can have **infinite depth**, hence may change monodromy of the parallel strands. The functor $F_{\mathbf{I}}$ is genuinely nontrivial on certain abstract curve objects containing $\mathscr{P}$. To finish the symmetry theorem for three-manifolds, HRW must prove that *no such peripheral component with any nonzero finite-rank local system* occurs in $\mathscr{C}(M)$.

8. The additional minus-type obstruction

The ordinary extended torus algebra used on Days 3–5 has a curved relation $\tilde{\partial}^2 = U$. That fact alone does **not** exclude all peripheral components. HRW22 invokes a stronger structure coming from a truncated minus version of bordered Floer theory.

8.1. A second U-variable

To avoid confusing two distinct roles, denote by **U** the truncation variable in HRW22's minus-type enhancement, reserving U for the curvature in the extended algebra. The relevant A_∞ algebra $\mathscr{A}^{\mathbf{U}}$ has operations μ_0, μ_2, and an additional μ_4, with $\mu_1 = \mu_3 = 0$. For cyclic indices $i \pmod 4$, the new four-input operation includes

$$\mu_4(\rho_{i+3}, \rho_{i+2}, \rho_{i+1}, \rho_i) = \mathbf{U}. \tag{8.1}$$

> **Theorem.** [SOURCE] HRW22, Theorem 41 (published numbering) A type D structure arising as the bordered invariant of a torus-boundary three-manifold is the restriction of a type D structure over the enhanced algebra $\mathscr{A}^{\mathbf{U}}$.

This theorem is an additional geometric input; it is not asserted for arbitrary algebraic extendable type D structures.

8.2. Extract the coefficient of U

The enhanced type D compatibility equation is

$$\mu_0 \circ \delta^0 + \mu_2 \circ \delta^2 + \mu_4 \circ \delta^4 = 0. \tag{8.2}$$

Take the coefficient of **U** (in the relevant truncation and reduced presentation). In the source paper this yields:

> **Theorem.** [SOURCE] HRW22, Corollary 42 (published numbering) If D'_0, D'_1, D'_2, D'_3 denote the length-one coefficient maps of the reduced extended type D structure of a manifold, then
>
> $$\boxed{\Psi' = \sum_{i=0}^{3} D'_i \circ D'_{i+1} \circ D'_{i+2} \circ D'_{i+3} = 0,} \tag{8.3}$$
>
> with subscripts taken modulo four.

The order of composition above is the one in HRW22; do not reverse it without changing the row/column convention. Formula (8.3) records the coefficient of the new **U** operation, and it is **not** simply the equation $M^2 = UI$ from the matrix normal form of Day 3.

8.3. Why the peripheral circle is impossible

Let $\mathscr{P}$ be the small immersed circle linking the puncture, equipped with a rank-r local system (V, Φ), where $\Phi \in GL(V)$. In its four-corner train-track presentation, the only possible four-step cyclic paths stay inside this component. The composition of the four consecutive coefficient maps is the parallel transport once around the peripheral circle. HRW computes, for a generator x on this component,

$$\boxed{\Psi'(x) = \Phi(x).} \tag{8.4}$$

But (8.3) requires $\Psi' = 0$, whereas Φ is invertible. Since $V \neq 0$, Φ cannot be the zero map. Contradiction.

> **Theorem.** [SOURCE] HRW22, Proposition 10 No component of the immersed-curve invariant of a torus-boundary three-manifold is a small circle linking the puncture.

The proof excludes **all** nonzero ranks and **all** invertible monodromies, not merely the trivial rank-one local system. It also explains why a seemingly harmless statement about $-I$ requires input beyond the hat type D structure's ordinary extendability.

8.4. A rank-two algebra check [DERIVED]

Let $V = \mathbb{F}_2^2$ and

$$\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{8.5}$$

Then $\det \Phi = 1$ and $\Phi^3 = I$. Therefore Φ is invertible and nonzero. A hypothetical peripheral summand with this monodromy would give $\Psi'|_V = \Phi \neq 0$, contradicting (8.3). This example shows why proving the exclusion only for identity monodromy would be insufficient.

9. Conjugation symmetry and genus-one mutation

Combining the two ingredients now finishes the proof. For all components of $\mathscr{C}(M)$, Proposition 10 rules out the sole exceptional peripheral type. Therefore $F_{\mathbf{I}}$ acts trivially on every actual manifold invariant. Equation (6.2), together with the LOT conjugation symmetry, yields the elliptic theorem.

Theorem. [SOURCE] HRW22, Theorem 7; Theorem 38; Corollary 39 For each $\mathfrak{s} \in \mathrm{Spin}^c(M)$,

$$\mathscr{C}(M, c(\mathfrak{s})) \simeq \epsilon\Big(\mathscr{C}(M, \mathfrak{s})\Big), \tag{9.1}$$

after applying the appropriate type A/type D and parametrization conventions. In particular, the *unlabeled* union $\mathscr{C}(M)$ is invariant under ϵ up to decorated regular homotopy.

Proof. [PROOF SKETCH] Logical completion LOT conjugation supplies $\widehat{CFD}(M, c(\mathfrak{s})) \simeq \mathbf{A} \boxtimes \widehat{CFD}(M, \mathfrak{s})$. HRW22, Proposition 40 identifies the geometric elliptic action with $\mathbf{E} \simeq \mathbf{A} \boxtimes \mathbf{I}$ (with compatible DA conventions). The coil reduction proves that $\mathbf{I}$ is invisible on nonperipheral components; Proposition 10 says the remaining peripheral possibility cannot occur in $\mathscr{C}(M)$. Therefore $\mathbf{E} \boxtimes \widehat{CFD}(M, \mathfrak{s}) \simeq \widehat{CFD}(M, c(\mathfrak{s}))$. The classification of decorated curves then gives (9.1). Summing over all spin-c structures yields the unlabeled symmetry.

9.1. Mutation

Let $Y = M_0 \cup_h M_1$ and define a **genus-one mutant** in this torus-gluing setting by $Y^\epsilon = M_0 \cup_{h \circ \epsilon} M_1$ (or by the equivalent convention on the other side of the gluing). The pairing theorem and equivariance under homeomorphisms give

$$\begin{aligned} \widehat{HF}(Y^\epsilon) &\cong HF\Big(\mathscr{C}(M_0), (h \circ \epsilon)\mathscr{C}(M_1)\Big) \\ &\cong HF\Big(\mathscr{C}(M_0), h\mathscr{C}(M_1)\Big) \\ &\cong \widehat{HF}(Y). \end{aligned} \tag{9.2}$$

Theorem. [SOURCE] HRW22, Corollary 9 $\widehat{HF}$ is invariant under genus-one mutation: the two closed manifolds obtained by regluing along the genus-one splitting with an elliptic involution have isomorphic hat Floer vector spaces.

Do not upgrade (9.2) to an assertion about *absolute Maslov gradings*, individual spin-c labels, or diffeomorphism types without separately checking those structures. The corollary, as cited, gives the ungraded vector-space isomorphism. The existence of mutant pairs that are not homeomorphic is entirely compatible with the result.

10. Worked computations

Example A. An explicit central involution

With T_a, T_b as in (2.1),

$$T_a T_b = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad (T_a T_b)^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad (T_a T_b)^3 = -I.$$

Thus six alternating elementary twists model a half-turn at the level of the marked torus mapping class group. For any slope vector $v = (p, q)$, $\epsilon(v) = (-p, -q)$ is the **same unoriented slope**, but its orientation has changed. This is why slope-only reasoning gives an incomplete proof of Theorem 7.

Example B. Filling two solid tori

Let M_0, M_1 both be solid tori. Suppose the meridian of M_0 is represented by $v = (1, 0)$ and the gluing sends the other meridian to $w = (2, 5)$. The two curve invariants are embedded meridian circles with trivial rank-one local systems. The five intersections are minimal and there are no immersed bigons between straight lifts. Hence

$$\dim \widehat{HF}(M_0 \cup_h M_1) = |\det(v, w)| = 5.$$

After precomposing the second marking with ϵ, w becomes $-w = (-2, -5)$ and the determinant changes sign but not absolute value. This calculation is a toy instance of (9.2). It is **not** a proof of genus-one mutation invariance for arbitrary immersed curves; that uses Theorem 7.

Example C. The matrix-level obstruction with nontrivial monodromy

Set $\Phi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Over $\mathbb{F}_2$,

$$\Phi^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Phi^3 = I.$$

For a hypothetical peripheral loop of rank two, the cyclic four-step coefficient operator is $\Psi' = \Phi$ on its fiber. Since $\Phi(1, 0)^T = (0, 1)^T \neq 0$, this violates the manifold relation $\Psi' = 0$. There is no need to know the precise eigenvalues or a Jordan form to obtain the contradiction.

Example D. Active/passive cancellation

Write the standard boundary coordinate as a column v. Change the *basis* by a matrix $S \in SL(2, \mathbb{Z})$, so new coordinates are $v' = S^{-1}v$. If an algebraic twist has induced coordinate action $v \mapsto Sv$, then the physical point described after changing the marking has coordinates $S^{-1}(Sv) = v$. This elementary cancellation is the linear shadow of HRW23, Proposition 5.1. The full theorem additionally tracks immersed corners and local systems.

11. Exercises with hints

Exercise. 1. Euclidean generation and the center Prove that the two elementary twist matrices in (2.1) generate $SL(2, \mathbb{Z})$ and calculate $(T_a T_b)^3$. Show directly that $-I$ is central. **Hint:** apply elementary operations to a primitive first column and use the determinant-one condition to finish.

Exercise. 2. Homology class versus decorated curve For $v = (2, 3)$ and $w = (1, 4)$ compute $i(v, w)$ and $i(v, -w)$. Explain why this equality fails to establish $\mathscr{C}(M) \simeq \epsilon\mathscr{C}(M)$ when the curve set has multiple self-intersecting components or local systems. **Hint:** distinguish homology class and regular-homotopy class of a decorated immersion.

Exercise. 3. Parametrization invariance Read HRW23, Proposition 5.1 and Figures 42–44. Reconstruct the passage from a ρ_2 edge to a ρ_{23} edge and from ρ_{12} to ρ_{123} under the twist bimodule. State precisely which unit arrows must be cancelled. Explain why both signs of the basis shear follow. **Hint:** move the crossover-arrow packet away from the short twist strip first.

Exercise. 4. Reflection versus half-turn Give explicit integral matrices for a reflection that fixes the longitude and for the elliptic involution. Compute their determinants and squares. Describe their effects on the orientation of the surface and on the ordered pair of meridian/longitude classes. **Hint:** take $R = \mathrm{diag}(-1, 1)$ and $\epsilon = -I$.

Exercise. 5. Why the abstract functors differ Using HRW22, Figures 27–32, identify the additional dashed operation of **A** not present in **E**. Explain how the small peripheral circle $\mathscr{P}$ witnesses the failure of $\mathbf{A} \simeq \mathbf{E}$ on the category of all abstract type D structures. Why does this not contradict Theorem 7? **Hint:** Theorem 7 is restricted to actual manifold invariants.

Exercise. 6. Derive the peripheral obstruction State the enhanced type D compatibility relation (8.2). Explain how the coefficient of **U** yields Corollary 42. For $\Phi \in GL_r(\mathbb{F}_2)$ prove that a peripheral summand with cyclic operator $\Psi' = \Phi$ is incompatible with $\Psi' = 0$. Test $r = 1$ and $r = 2$ explicitly. **Hint:** any invertible matrix on a nonzero space is nonzero.

Exercise. 7. Genus-one mutation and pairing Assume the elliptic invariance theorem. Derive $\widehat{HF}(M_0 \cup_{h\circ\epsilon} M_1) \cong \widehat{HF}(M_0 \cup_h M_1)$ using only the decorated-curve pairing and its homeomorphism covariance. Identify which part is not proved by an intersection determinant. **Hint:** substitute $\epsilon\mathscr{C}(M_1) \simeq \mathscr{C}(M_1)$ before taking Floer homology.

Exercise. 8. Spin-c labels and orientation Explain why Theorem 7's unlabeled invariance need not identify each $\mathrm{Spin}^c(M)$ summand with itself. Relate the two summands indexed by $\mathfrak{s}$ and $c(\mathfrak{s})$. Say exactly what additional information is needed to compare their relative Maslov gradings. **Hint:** review Day 8 and HRW22, §2.

12. Solutions and proof checkpoints

Exercise 1. The Euclidean proof is given in §2. Multiply $T_a T_b = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ three times to obtain $-I$; a scalar matrix commutes with every matrix. This does not mean that ϵ is isotopic to the identity *relative to the puncture*.

Exercise 2. $\det(v, w) = 2 \cdot 4 - 3 \cdot 1 = 5$; replacing w by $-w$ changes the determinant to -5 but leaves $|\det| = 5$. An arbitrary immersed curve may have extra intersections and differential bigons, while monodromy may change their counts. The determinant gives only minimal intersections for embedded primitive slopes.

Exercise 3. In HRW23's chosen basis conventions, the extra new ι_1 switch gives a ρ_2 segment, while an old ρ_2 ending at a ι_0 switch can be rerouted to a new ρ_{23} ending at the new switch; similarly ρ_{12} becomes ρ_{123}. The extra $\rho_\varnothing$ contributions resulting from ρ_3 edges must be cancelled. To complete the exercise, check all local cases against Figure 44; the present lecture deliberately does not claim a figure-free derivation.

Exercise 4. $R = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ has determinant -1, while $\epsilon = -I$ has determinant $+1$. Both square to I, but only the second preserves orientation of the boundary. A reflection and a half-turn encode different three-manifold operations.

Exercise 5. The dashed higher operation shown in HRW22, Figure 27 is present in the conjugation algebra bimodule **A** and absent in the elliptic bimodule **E**. On $\mathscr{P}$ this difference can survive cancellation and even change local-system data. Theorem 7 is not contradicted because Proposition 10 proves no peripheral component appears in the invariant of an actual manifold.

Exercise 6. The enhanced equation has $\mu_0 + \mu_2\delta^2 + \mu_4\delta^4 = 0$ after inserting the structure maps and suppressing identities. The **U**-term gives $\Psi' = 0$. For $r = 1$, the only invertible 1×1 matrix over $\mathbb{F}_2$ is (1), not zero. For $r = 2$, Φ in (8.5) has determinant 1, so is nonzero. The same contradiction works for every $r > 0$.

Exercise 7. Using (4.2),

$$\begin{aligned} \widehat{HF}(M_0 \cup_{h\circ\epsilon} M_1) &\cong HF(\mathscr{C}(M_0), h(\epsilon\mathscr{C}(M_1))) \\ &\cong HF(\mathscr{C}(M_0), h\mathscr{C}(M_1)) \\ &\cong \widehat{HF}(M_0 \cup_h M_1). \end{aligned}$$

The middle isomorphism uses elliptic invariance of **decorated curves**, not determinant arithmetic.

Exercise 8. Conjugation acts on the spin-c indexing set, so $\mathscr{C}(M, \mathfrak{s})$ can map to $\mathscr{C}(M, c(\mathfrak{s}))$ with $c(\mathfrak{s}) \neq \mathfrak{s}$. Relative Maslov comparisons require the grading cover, compatible lifts, and (for multicomponent curve sets) phantom grading arrows, not just the unlabeled immersed picture.

Proof checkpoints

- □ Derive $(T_a T_b)^3 = -I$ and explain why the elliptic involution is nontrivial relative to the puncture.

- ☐ State both basis-shear equivalences of HRW23, Proposition 5.1, without reversing active/passive conventions.
- ☐ Explain why HRW23's original gluing map uses $\overline{h}$, and how HRW22 permits h instead.
- ☐ Distinguish orientation reversal of M from spin-c conjugation.
- ☐ State the roles of **A**, **E**, **I** and why $\mathbf{A} \neq \mathbf{E}$ on all abstract modules.
- ☐ Explain finite-depth coil cancellation and why $\mathscr{P}$ is exceptional.
- ☐ State HRW22, Theorem 41 and Corollary 42, and explain why the latter rules out peripheral summands of any rank.
- ☐ Derive Corollary 9 (genus-one mutation invariance) directly from elliptic invariance and the pairing theorem.

13. Oral examination with model answers

Question. 1. Why does parametrization independence reduce to two elementary moves? Because the mapping class group of an oriented marked torus acts through $SL(2,\mathbb{Z})$, generated by elementary Dehn twists. HRW23, Proposition 5.1 proves compatibility of the immersed-curve construction with those generators, including the action on local systems; compositions give arbitrary changes of basis.

Question. 2. Does $-I$ act trivially on every curve in a punctured torus? No. Its action is nontrivial on the punctured surface and on arbitrary abstract decorated curve objects. HRW22 proves it preserves the *special curve sets arising from three-manifolds*. The restriction is essential; the abstract peripheral-loop example exposes the difference.

Question. 3. Why did the 2023 gluing theorem contain an elliptic involution? The algebraic pairing requires converting one input to a type A structure, represented geometrically using a reflection across the anti-diagonal. In the boundary coordinates fixed by the gluing convention, that reflection corresponds to composing the gluing map with the elliptic involution. The later elliptic symmetry removes it.

Question. 4. What is the key obstruction to identifying conjugation and elliptic bimodules everywhere? **A** and **E** differ by a higher operation. The half-identity correction can change a peripheral loop's local system because its new arrows may have infinite depth. Ordinary arrow cancellation fails there, so the two functors need not agree on the full abstract category.

Question. 5. Why are small circles around the puncture forbidden for actual manifold invariants? The enhanced minus-type bordered structure supplies a relation $\Psi' = \sum_i D'_i D'_{i+1} D'_{i+2} D'_{i+3} = 0$ (with the appropriate composition order). On a puncture-linking component, this operator is the invertible local-system monodromy. Zero cannot equal an invertible endomorphism of a nonzero fiber.

Question. 6. Why does genus-one mutation preserve hat Floer homology? The geometric pairing computes the Floer homology of the glued manifold. Reguling by ϵ replaces one curve set by its elliptic image; elliptic invariance identifies it with the original decorated curve set. The resulting intersection Floer homologies, and therefore the closed-manifold hat groups, agree as vector spaces.

Question. 7. Does the mutation theorem preserve absolute gradings or spin-c labels? Not from Corollary 9 alone. The stated ungraded isomorphism is enough to conclude the vector-space invariant agrees. A spin-c or Maslov-refined assertion requires compatible identifications of grading data; these are separate issues from the unlabeled elliptic symmetry.

14. Reading assignments and references

Minutes	Close reading / written task	Deliverable
40	HRW23 §5 opening and Proposition 5.1, pp. 52–53	State both shear equivalences with correct signs
50	HRW23, proof of Proposition 5.1, Figures 42–44, pp. 53–55	Reconstruct the twist-bimodule/curve local cases

	Minutes	Close reading / written task	Deliverable
	35	HRW23, end of §5 and Remark 5.2, pp. 55–56	Derive the original $\bar{h}$ pairing
	35	HRW22 §3.1 and §3.2 through Proposition 40	Distinguish reflection, conjugation, and elliptic action
	55	HRW22 §3.2, Figures 27–34, Theorem 38 proof	Explain coils and the exception $\mathscr{P}$
	35	HRW22 Theorem 41, Corollary 42, Proposition 10	Derive and interpret the $\Psi' = 0$ obstruction
	40	Exercises 1–8 and oral rehearsal	Solve at least 1, 4, 6, 7 in full

Total: 290 minutes = 4 h 50 min. Suggested Day 10 continuation: HRW22 §4, *Knot Floer homology*, including the LOT algorithm from $CFK^-(K)$, the recovery of core-knot $\widehat{HFK}$ from noncompact slope lines, and the right-handed trefoil worked example.

Primary bibliographic references

[HRW23] J. Hanselman, J. Rasmussen, L. Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, Geom. Topol. 27 (2023), 823–924. See Theorems 1.1–1.2 and §5, especially Proposition 5.1, Figures 42–44, Remark 5.2. arXiv:1604.03466v3.

[HRW22] J. Hanselman, J. Rasmussen, L. Watson, *Heegaard Floer homology for manifolds with torus boundary: properties and examples*, Proc. Lond. Math. Soc. 125 (2022), 879–967. See §3.1–3.2, Theorem 7, Corollary 9, Proposition 10, Theorem 38, Corollary 39, Proposition 40, Theorem 41, Corollary 42, Figures 25–34 (numbering may differ between the journal and the arXiv preprint). Open-access article (PMC) · arXiv:1810.10355v1.

[LOT, supporting] R. Lipshitz, P. Ozsváth, D. Thurston, *Bordered Heegaard Floer Homology*, Mem. Amer. Math. Soc. 254 (2018), no. 1216. The mapping-class DA bimodules and the box-tensor pairing are used as inputs rather than reproved in this lecture. DOI.