Day 10 — Knot Floer Complexes, Immersed Curves, and Filling Cores

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Primary papers: HRW22, published article, §4 and HRW23, main article.
PDF companion: HRW_Day10.pdf

> **WARNING: Source and convention policy** We write $\mathscr{C}(M) = \widehat{HF}(M)$ for the **decorated immersed curves of a torus-boundary manifold**, never for the closed-manifold vector space. Throughout $\mathbb{F} = \mathbb{F}_2$. For the coefficient of the *knot Floer* differential, use U; the curvature parameter of Day 3 is a distinct object even when also printed U. Published HRW22 theorem numbers (rather than the 2018 preprint numbering) are used: **Proposition 46, Proposition 48, Theorem 52, Remark 53, Proposition 56**. Results proved independently below are labelled *derived*. Figure-dependent arguments are explicitly marked **proof sketch**.

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1. The two directions and the critical logical asymmetry

Let $K \subset S^3$, $M_K = S^3 \smallsetminus \nu K$, and write μ, λ for the preferred meridian and Seifert longitude. The two ways to pass between knots and bordered curves are genuinely different:

$$CFK^-(K) \xrightarrow{\text{LOT}} \widehat{CFD}(M_K, \mu, \lambda) \xrightarrow{\text{HRW23}} \mathscr{C}(M_K), \tag{1.1}$$

while, for **any** torus-boundary M and primitive filling slope μ with filling core $K_\mu \subset Y = M(\mu)$,

$$\mathscr{C}(M) \xrightarrow{\text{pair with a meridional arc } L_\mu} \widehat{HFK}(Y, K_\mu). \tag{1.2}$$

> **THEOREM: [SOURCE: HRW22, introductory Theorem 11]** For a knot $K \subset S^3$, its knot-complement immersed-curve invariant $\mathscr{C}(M_K)$ is determined by $CFK^-(K)$. This is the composition of the LOT construction with the HRW immersed-curve classification.

> **THEOREM: [SOURCE: HRW22, introductory Proposition 12]** Fix $\mathfrak{s} \in \mathrm{Spin}^c(M)$ and the corresponding relative Spin-c label $\bar{\mathfrak{s}}$ of the filling core. Then
>
> $$\widehat{HFK}(K_\mu, \bar{\mathfrak{s}}) \cong HF\big(\mathscr{C}(M, \mathfrak{s}), L_{\mu,\bar{\mathfrak{s}}}\big). \tag{1.3}$$
>
> The arc L_μ is **noncompact**, of slope μ, and approaches the removed puncture at its two ends.
>
> This is not the same test object as the *closed* slope curve used to compute $\widehat{HF}(M(\mu))$.

> **WARNING: The converse of Theorem 11 is not stated** HRW22 proves that the **associated graded** knot Floer complexes are recovered from curves. It does *not* claim that the bare hat-type curve set canonically reconstructs every diagonal term of the full filtered differential of CFK^-. In §4.3 it explicitly warns that an unrestricted two-basepoint bigon count may fail to square to zero. The stronger theory of fully decorated curves with bounding chains is **not** the same invariant used in these two HRW papers.

We shall make this logical asymmetry computationally visible in the trefoil example.

2. Filtration, basepoints, and three distinct homologies

Let (Y, K) be a null-homologous knot in an oriented 3-manifold and choose a doubly pointed Heegaard diagram with basepoints w, z. In the conventions of HRW22, §4.1, the minus knot Floer complex is a free finite-rank $\mathbb{F}[U]$ chain complex, filtered by Alexander grading:

$$\dots \subset \mathscr{F}_{a-1}C \subset \mathscr{F}_a C \subset \dots, \qquad C = CFK^-(Y,K), \qquad A(Ux) = A(x) - 1. \tag{2.1}$$

A holomorphic disk class $\phi \in \pi_2(x,y)$ contributes $U^{n_w(\phi)}y$ (when its expected dimension and the usual conditions are satisfied), and

$$A(x) - A(y) = n_z(\phi) - n_w(\phi) \quad \text{for the *unshifted* generator } y. \tag{2.2}$$

Therefore

$$A(U^{n_w(\phi)}y) = A(y) - n_w(\phi) = A(x) - n_z(\phi). \tag{2.3}$$

Every differential term is filtration non-increasing. A disk with $n_z = 0$ preserves Alexander filtration; a disk with $n_z > 0$ lowers it.

Three complexes must be kept separate.

Object	Differential retains	Homology
CFK^-	all counted disks (with U^{n_w})	$HF^-(Y)$ as an unfiltered module
$gCFK^- = \mathrm{gr}_A\, CFK^-$	only $n_z = 0$	$HFK^-(Y,K)$
$g\widehat{CFK} = gCFK^-/(U)$	only $n_z = n_w = 0$	$\widehat{HFK}(Y,K)$

The notation $\widehat{CFK} = CFK^-/(U)$ **without taking the associated graded** instead computes $\widehat{HF}(Y)$ on homology, not $\widehat{HFK}(Y,K)$ in general.

> **PROOF: Derived lemma: associated graded of a filtered chain complex** Suppose a differential d preserves an increasing filtration $\mathscr{F}_a C$. Then its leading filtration-preserving part d_0 satisfies $d_0^2 = 0$. Indeed, $d^2 = 0$ and taking the induced map on every quotient $\mathscr{F}_a/\mathscr{F}_{a-1}$ yields $[d]^2 = [d^2] = 0$. In the knot Floer interpretation this d_0 counts exactly the disks with $n_z = 0$. **This proof applies to the genuine CFK^- differential, not automatically to the naive curve-defined C^- considered in §7 below.**

The Maslov rules in the S^3 normalization are $M(Ux) = M(x) - 2$ and $M(dx) = M(x) - 1$. The Alexander grading is a filtration on the whole complex but an honest grading on its associated graded.

3. Simultaneously simplified bases and the curve algorithm

3.1. What the two simplified conditions mean

A **vertically simplified** filtered basis $\{\xi_i\}$ organizes the differential modulo U: its nonzero leading vertical terms pair generators as $\xi_i \mapsto \xi_j \pmod U$. Thus a vertical arrow from x to y has $n_w = 0$, $n_z > 0$, and length

$$\ell_v = A(x) - A(y) > 0.$$

A **horizontally simplified** basis $\{\eta_i\}$ organizes the differential in the associated graded $gCFK^-$: a horizontal arrow $x \to y$ contributes $U^{\ell}y$ with

$$\ell_h = A(y) - A(x) > 0; \qquad n_z = 0, \quad n_w = \ell_h.$$

Each basis exists individually. The paper does not assume one basis is simultaneously simplified in the general case.

> **THEOREM: [SOURCE: HRW22, Proposition 46] If** $CFK^-(K)$ admits a single basis that is both horizontally and vertically simplified, then its complement M_K is **loop type** (its immersed curves can be taken with trivial rank-one local systems).

PROOF: Proof (the algebraic step in HRW22) The LOT algorithm associates to each basis element of CFK^- one ι_0 generator of $\widehat{CFD}(M_K)$. Each vertical arrow gives a chain of ι_1 generators on one side, and each horizontal arrow gives such a chain on the other. The two unmatched generators are connected by an **unstable chain**. In a common simplified basis every ι_0 vertex has exactly two incident chains (counting both ends of one chain when applicable). The resulting type D graph is valence two: this is the loop-type criterion. A fully explicit check of the intermediate Reeb-chord labels is the supporting LOT Chapter 11 calculation; HRW22 gives the valence-two argument directly.

WARNING: Do not reverse the implication Proposition 46 gives a **sufficient condition**, not a characterization of every knot complex. HRW22, Remark 47 notes that the existence of a simultaneous basis for every knot was not established there. The general construction uses a *pair* of simplified bases and the crossover-arrow calculus.

3.2. The cylinder and four-step recipe

Take the covering cylinder

$$\overline{T}_M = \left([-\tfrac{1}{2}, \tfrac{1}{2}] \times \mathbb{R}\right)/\left((-\tfrac{1}{2}, t) \sim (\tfrac{1}{2}, t)\right), \tag{3.1}$$

with horizontal direction the Seifert longitude λ, vertical direction the meridian μ, and lifts of the puncture at $(0, n + \frac{1}{2})$ for $n \in \mathbb{Z}$.

THEOREM: [SOURCE: HRW22, Proposition 48] Assume a common horizontally and vertically simplified basis. Construct $\mathscr{C}(M_K)$ in the cylinder as follows: 1. For each basis generator x, draw a short horizontal segment at height $A(x)$. 2. For each vertical arrow $x \to y$, connect the **left** ends of their segments by an arc. 3. For each horizontal arrow $x \to y$, connect the **right** ends by an arc. 4. Exactly one left end and one right end remain unmatched. Connect them around the cylinder, using the boundary points $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$.

The short segments have horizontal extent $[-1/4, 1/4]$ in the paper. Their heights, the locations of the punctures, and which ends are attached are **not arbitrary drawing choices**: the conventions determine the Alexander labels and the unstable chain.

PROOF: Proof of Proposition 48 (HRW22, compressed but complete given the LOT dictionary) The LOT conversion sends each CFK^- generator to an ι_0 generator. A vertical arrow of length ℓ_v becomes a type D chain with ℓ_v intermediate ι_1 generators; the corresponding train-track segment moves downward by ℓ_v to join the **left** ends. The horizontal arrow of length ℓ_h gives the analogous chain moving upward to join the **right** ends. The two distinguished unmatched generators are joined by the unstable chain; this becomes the single path winding once across the horizontal identification of the cylinder. Since the bordered graph is valence two, these arcs join to a union of immersed closed curves. The local systems are trivial in this presentation. This reproduces the four steps. The specific Reeb-chord chain diagrams are in HRW22, proof of Proposition 48, and LOT18 Chapter 11.

This is a source theorem, not a universal recipe for a complex with unsimplified diagonal terms and arbitrary basis.

4. Trefoil computation from beginning to end

4.1. A model CFK^- for the right-handed trefoil

Consider the rank-three filtered $\mathbb{F}[U]$ complex with generators a, b, c:

$$A(a) = 1, \quad A(b) = 0, \quad A(c) = -1; \qquad \partial a = \partial c = 0, \quad \partial b = Ua + c. \tag{4.1}$$

We choose Maslov gradings

$$M(a) = 0, \quad M(b) = -1, \quad M(c) = -2.$$

The two summands of ∂b both have Maslov grading -2, one below $M(b)$. The differential squares to zero because a, c are cycles. The coefficient U lowers Alexander grading by one.

This is the standard three-generator staircase model of $CFK^-(T(2,3))$ with one choice of grading and arrow orientation; it is **a filtered chain model**, not a claim that its three generators are all homology classes of CFK^-.

4.2. Verify the two lengths and their basepoints

The term $b \mapsto c$ has coefficient 1: $n_w = 0$, and

$$A(b) - A(c) = 0 - (-1) = 1 = n_z.$$

It is a **vertical** arrow of length one, hence joins **left** ends of the height-zero and height-minus-one segments.

The term $b \mapsto Ua$ has $n_w = 1$ and $n_z = 0$: indeed,

$$A(b) - A(a) = -1 = n_z - n_w.$$

It is a **horizontal** arrow of length one, hence joins **right** ends of the height-zero and height-plus-one segments.

There are three horizontal pieces at heights $1, 0, -1$; the left pair is $b \leftrightarrow c$, and the right pair is $b \leftrightarrow a$. The unmatched **left** end belongs to a, and the unmatched **right** end belongs to c. The unstable arc connects these through the cylindrical identification.

This is the combinatorial description of the trefoil curve depicted in HRW22, Figure 3 and HRW23, Example 6.8 / Figure 47. The precise smooth isotopy should be compared with those diagrams, rather than inferred from a schematic algebra graph.

4.3. Compute three different homologies

Total minus complex. The relation imposed by $\partial b = Ua + c$ is $c = Ua$ in homology. There is no nonzero cycle involving b over the domain $\mathbb{F}[U]$. Thus

$$H_*(CFK^-) = \mathbb{F}[U]\langle [a] \rangle \cong HF^-(S^3). \tag{4.2}$$

Alexander associated graded. The term $b \to c$ drops Alexander filtration, so it is removed. The filtration-preserving differential is

$$\partial_{\mathrm{gr}} b = Ua, \qquad \partial_{\mathrm{gr}} a = \partial_{\mathrm{gr}} c = 0.$$

Consequently

$$HFK^-(T(2,3)) \cong \mathbb{F}[U]\langle [c] \rangle \oplus \Big(\mathbb{F}[U]/(U)\Big)\langle [a] \rangle, \tag{4.3}$$

with the gradings inherited from (4.1). The first summand is the free tower; the second is U-torsion. Notice that $[b]$ is not an associated-graded homology class over $\mathbb{F}[U]$.

Hat associated graded. Reduce the *associated graded* complex modulo U. Both arrows disappear, yielding

$$\widehat{HFK}(T(2,3), A) \cong \begin{cases} \mathbb{F}, & A \in \{-1, 0, 1\}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

The Maslov grades are respectively $-2, -1, 0$. Hence

$$\sum_{A,M} (-1)^M \dim \widehat{HFK}_M(K, A) t^A = t^{-1} - 1 + t. \tag{4.5}$$

This is the Alexander polynomial of the right-handed trefoil in the symmetric normalization.

> **WARNING: An excellent oral-exam trap** If you instead set $U = 0$ **before** taking the Alexander associated graded, the complex has $\partial b = c$ and one-dimensional homology, namely $\widehat{HF}(S^3)$. This is not the three-dimensional $\widehat{HFK}(T(2,3))$. Filtration, basepoints, and the two quotient operations cannot be silently interchanged at the level of differentials.

4.4. Read g, τ, and ϵ geometrically

The maximum and minimum discrete heights of the tight trefoil curve are 1 and -1, so the half-range is 1, matching $g(T(2,3)) = 1$. The unique segment wrapping nonvertically around the cylinder identifies the distinguished vertical-homology generator at height $\tau = 1$. Its next turn is upward under HRW's rightward convention, so $\epsilon = 1$. Hence the slope of that segment is

$$2\tau - \epsilon = 2(1) - 1 = 1. \tag{4.6}$$

The geometric interpretations are HRW22, discussion after Proposition 48; the numeric evaluation is ours. This rule applies to the normal-form peg-board convention, not to an arbitrarily sheared drawing.

5. When simplified bases differ: crossover arrows

For a general knot, choose a vertically simplified basis $\{\xi_0, \xi_1, \dots, \xi_{2n}\}$ and a (possibly different) horizontally simplified basis $\{\eta_0, \eta_1, \dots, \eta_{2n}\}$. The LOT type D description contains paired vertical chains in the ξ basis, paired horizontal chains in the η basis, and one unstable chain connecting ξ_0 to η_0.

After passing to the hat type D module, the allowable filtered change of basis has coefficients reduced modulo U:

$$\widehat{\xi}_i = \sum_j \hat{a}_{ij}\widehat{\eta}_j, \qquad \hat{a}_{ij} \in \mathbb{F}, \quad \hat{a}_{ij} = 0 \text{ unless } A(\xi_i) = A(\eta_j). \tag{5.1}$$

Thus the nontrivial ordinary basis changes occur **within one Alexander level**. Elementary changes of basis $\widehat{\xi}_i \mapsto \widehat{\xi}_i + \widehat{\xi}_j$ are represented by crossover arrows at the two ends of the corresponding horizontal strands. This is HRW22, §4.2, Figures 39–40.

5.1. A matrix calculation

Let E_{12} be an elementary matrix unit and set $P = I + E_{12}$ over $\mathbb{F}_2$. Then $P^{-1} = P$. Transporting a differential matrix D by this basis change gives

$$D' = PDP^{-1} = D + E_{12}D + DE_{12} + E_{12}DE_{12}. \tag{5.2}$$

The last term can be nonzero; it is the algebraic reflection of paths that enter and leave the crossover arrow. By construction,

$$(D')^2 = PD^2P^{-1} = 0.$$

> **WARNING: No blanket 'all knot complements are loop type' conclusion** A single common simplified basis makes the valence-two graph immediate. Without it, HRW22 uses the **general** arrow-reduction algorithm of HRW23, §§3.4–3.7. Extra crossover arrows might initially remain; the structure theorem allows local systems. The paper does not authorize discarding them by fiat.

5.2. A practical six-step construction [HRW22, §4.2]

Start with any filtered basis: draw one horizontal segment at each generator's Alexander height; place oriented vertical- and horizontal-arrow arcs on the left and right; slide their endpoints until the track is arcs plus crossover arrows; attach the single pair of free ends across the cylinder; finally simplify using the HRW23 train-track algorithm. HRW22 explicitly flags the step of pairing all extra directed edges into crossover arrows as not proved geometrically in that shortcut; its validity is explained by returning to the construction using two simplified bases.

In the connected-sum example $T(2,3)\#T(2,3)$, HRW22, Figures 38–40, show a basis change that introduces crossover arrows before simplification. The tensor-product basis for the two trefoils is *not* a license to regard every picture as an already reduced multicurve.

6. Filling-core knot Floer homology from a curve

Let M be any compact torus-boundary manifold, let μ be a primitive Dehn filling slope, and set

$$Y = M(\mu), \qquad K_\mu = \text{the core of the attached solid torus.}$$

Replace the original puncture by a small marked disk D with two basepoints w, z. Choose a **closed representative** of μ passing through D and separating w, z. Transverse intersections of this representative with $\mathscr{C}(M)$ form the generators of a candidate $\mathbb{F}[U]$ module.

For an immersed bigon $B : x \to y$, write $n_w(B), n_z(B)$ for its basepoint multiplicities. Formally set

$$d_{\text{naive}} x = \sum_y \sum_{B: x \to y} U^{n_w(B)} \, \mathrm{PT}_B(y), \tag{6.1}$$

where PT_B includes local-system parallel transport; coefficients are mod two. The Alexander filtration is determined by

$$A(x) - A(y) = n_z(B) - n_w(B), \qquad A(Ux) = A(x) - 1. \tag{6.2}$$

The **proven** differential keeps only $n_z(B) = 0$; call its complex $gC^-(\mathscr{C}(M), \mu)$. Reducing further modulo U gives $g\widehat{C}(\mathscr{C}(M), \mu)$, which counts only $n_z = n_w = 0$.

> **THEOREM: [SOURCE: HRW22, Theorem 52; Theorem 51 in arXiv:1810.10355v1]** For the filling core $K_\mu \subset Y = M(\mu)$,
>
> $$gC^-(\mathscr{C}(M), \mu) \simeq gCFK^-(Y, K_\mu), \qquad g\widehat{C}(\mathscr{C}(M), \mu) \simeq g\widehat{CFK}(Y, K_\mu), \tag{6.3}$$
>
> as the stated filtered chain homotopy types. Thus their homologies compute HFK^- and $\widehat{HFK}$, respectively.

6.1. Why the theorem is true: the bordered module X

HRW22 reduces (6.3) to the LOT knot Floer pairing. The relevant right type A module X has a single generator x_0 in idempotent ι_0 with operations

$$m_{3+i}(x_0, \rho_3, \underbrace{\rho_{23}, \dots, \rho_{23}}_{i \text{ times}}, \rho_2) = U^{i+1} x_0, \qquad i \geqslant 0. \tag{6.4}$$

An intersection of $\mathscr{C}(M)$ with the meridian line corresponds to an ι_0 generator of $\widehat{CFD}(M, \mu, \ell)$, where $\ell \cdot \mu = 1$. A bigon avoiding z and meeting w with multiplicity $i+1$ corresponds to a type D path labelled

$$\underbrace{\rho_3, \rho_{23}, \dots, \rho_{23}}_{i}, \rho_2.$$

The box tensor product $X \boxtimes \widehat{CFD}$ weights this path by U^{i+1}, exactly as in the curve count. Paths and local-system transport agree by the established train-track dictionary (HRW23, §4). Thus the two differentials agree on generators, establishing (6.3) after the LOT identification.

> **PROOF: Proof-status disclosure** The displayed correspondence of algebraic words with bigons and the U-powers is the central combinatorial part of HRW22's **short proof** of Theorem 52. Its input that the bordered box product computes the knot Floer complex is LOT18, Theorem 11.19 and Lemma 11.20; those foundational analytic results are cited, not reproved. The proof is complete **relative to those source theorems**.

6.2. Hat pairing with a noncompact arc

Instead of using a closed meridional line and two basepoints, remove the marked disk and regard the meridional test object as a proper arc L_μ in the once-punctured torus, with ends asymptotic to the puncture. By HRW22, Remark 53,

$$\boxed{\widehat{HFK}(Y, K_\mu) \cong HF(\mathscr{C}(M), L_\mu).} \tag{6.5}$$

This is precisely the refined assertion of introductory Proposition 12 once the relative Spin-c labels are specified.

7. The graded theorem and the dangerous non-theorem

The distinction here is perhaps the most important point in today's lecture.

> **WARNING: [SOURCE: HRW22, immediately before Theorem 52]** In general one **does not expect** $d^2_{\text{naive}} = 0$ on the unrestricted curve-defined module $C^-(\mathscr{C}(M), \mu)$ from (6.1). The construction need not define a chain complex. One must not present its homology as a theorem of HRW22.

Why is that warning compatible with the fact that genuine CFK^- always has a square-zero differential? The bare immersed curves encode a **hat-type bordered object**. Counting all bigons weighted only at w may lack the extra obstruction/bounding-chain data necessary for the usual compactification and cancellation of broken configurations. Keeping only $n_z = 0$, however, is justified by the actual LOT pairing and yields a genuine differential. HRW22, §8 discusses speculative minus-type upgrades; a later theory with additional bounding-chain data is a different construction.

7.1. The trefoil as an algebraic laboratory

In (4.1) the two arrows are

$$b \xrightarrow{n_w=0,n_z=1} c, \qquad b \xrightarrow{n_w=1,n_z=0} a.$$

The $n_z = 0$ arrow survives in $gCFK^-$, and **both** disappear in $g\widehat{CFK}$. The diagonal-free trefoil happens to admit a complete filtered differential represented by these two arrows. This agreement in a simple example must not be promoted to a general recovery result.

7.2. A precise lemma for graded counts

> **THEOREM: [DERIVED]** A bigon of w-multiplicity p and z-multiplicity q contributes an arrow $x \to U^p y$ of Alexander filtration drop exactly q. In particular $q = 0$ characterizes the associated-graded differential.
>
> **PROOF** By (6.2), $A(y) = A(x) - q + p$. Therefore $A(U^p y) = A(y) - p = A(x) - q$. The drop is q. This does **not** prove existence of a holomorphic bigon or a cancellation theorem: it identifies the filtration degree of an already admissible contribution.

8. Relative Spin-c structures and Alexander heights

A useful refinement in HRW22, §4.4 avoids flattening all Alexander gradings into one unlabeled vector space. For a slope μ, let γ_μ denote the two parallel sutures on ∂M. The set $\mathrm{Spin}^c(M, \gamma_\mu)$ is a torsor over $H_1(M)$, and the fiber over $\mathfrak{s} \in \mathrm{Spin}^c(M)$ is an H_M-torsor, where

$$H_M = \mathrm{im}\Big(H_1(\partial M) \to H_1(M)\Big).$$

The same group acts on lifts of L_μ to the grading cover $\overline{T}_{M,\mathfrak{s}}$ (HRW23, §6.2). Thus a relative Spin-c class $\bar{\mathfrak{s}}$ selects a particular lift $L_{\mu,\bar{\mathfrak{s}}}$.

Choose a relative homology class $[\Sigma] \in H_2(M, \partial M)$ with $\partial[\Sigma] = \lambda$ (the Seifert longitude). Define

$$A(\bar{\mathfrak{s}}) = \frac{1}{2}\langle c_1(\bar{\mathfrak{s}}), [\Sigma]\rangle. \tag{8.1}$$

If we shift a relative Spin-c class by $a \in H_M$, then

$$A(\mathfrak{s} + a) - A(\mathfrak{s}) = a \cdot \lambda. \tag{8.2}$$

The same shift changes the vertical height of the corresponding lifted meridional test arc by $a \cdot \lambda$. Thus the two gradings agree after choosing one additive normalization.

THEOREM: [SOURCE: HRW22, Proposition 56; compare HRW23, Example 6.8] After fixing the height normalization, the Alexander grading of $\mathfrak{s}$ equals the height of the midpoint of the corresponding lift $L_{\mu,\mathfrak{s}}$.

8.1. Proof of the affine-height statement

PROOF The first Chern class satisfies $c_1(\mathfrak{s}+a) = c_1(\mathfrak{s})+2\,\mathrm{PD}(a)$. Pairing with $[\Sigma]$ and dividing by two gives (8.2). The height change under deck translation by a is the oriented intersection $a \cdot \lambda$, exactly the same expression. Thus the difference between the Alexander function and the geometric height is constant on the H_M-torsor; normalize at one base lift to make it zero everywhere. This is HRW22's argument immediately before Proposition 56.

For a knot $K \subset S^3$, $H_1(M_K) \cong \mathbb{Z}$ and the Seifert longitude vanishes in $H_1(M_K)$, so the cover is an infinite vertical cylinder. The meridian lifts are naturally indexed by integer Alexander levels after the usual normalization.

8.2. Trefoil check and the longitude check

For the right-handed trefoil, HRW23, Example 6.8 / Figure 47, shows meridional lifts $L_{\mu,-1}, L_{\mu,0}, L_{\mu,1}$, each contributing one hat knot Floer class, and no other meridional lifts contributing. This agrees with (4.4).

By contrast, the **longitude** test arcs $L_{\lambda,t}$ in HRW22, introductory discussion of Figure 3, have

$$HF(\mathscr{C}(M_K), L_{\lambda,t}) \cong \begin{cases} \mathbb{F}^2, & t = \pm\frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{8.3}$$

The half-integral labels and the ranks two are a useful check that the filling slope changes both the family of test arcs and the relevant relative Spin-c torsor. They should not be conflated with the meridional Alexander gradings $-1, 0, 1$.

9. A second calculation: the unknot and filling cores

Let $M = D^2 \times S^1$ be a solid torus and denote by $\alpha \in H_1(\partial M)$ the slope bounding a disk in M. Its curve invariant is the single embedded circle of slope α, with trivial rank-one local system (HRW23, §1.2).

9.1. The unknot

For the unknot in S^3, take $CFK^- = \mathbb{F}[U]\langle u \rangle$ with $A(u) = 0$ and $\partial u = 0$. Proposition 48 gives one horizontal segment at height zero, no left or right arrows, and an outside connection closing to an essential curve. Pairing with the meridional proper arc has a single transverse intersection and no differential. Thus

$$\boxed{\widehat{HFK}(\text{unknot}, 0) = \mathbb{F}.}$$

The full minus homology is the one free tower $\mathbb{F}[U]$.

9.2. A lens-space surgery core [derived application]

Take $\alpha = (1, 0)$ and a primitive filling slope $\mu = (2, 5)$ on the boundary of the solid torus, in a basis where geometric intersection is the absolute determinant. Then

$$i(\alpha, \mu) = \left|\det \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}\right| = 5.$$

Straight representatives of α and the test arc L_μ can be chosen in minimal position. They have five transverse intersections; they cannot bound an immersed puncture-free bigon in the corresponding geodesic/minimal-position configuration. Hence

$$\dim_{\mathbb{F}} \widehat{HFK}(K_\mu) = 5.$$

This is the simple core knot in a lens-space filling (the gluing determinant gives $|H_1(Y)| = 5$). In the Spin-c-refined picture, the five intersections represent five different relative deck classes. This rank calculation is an application of the source pairing, not a separate statement of HRW22 that all fillings produce five classes.

10. Proof and paper dependency map

The dependencies for the two directions should be stated precisely in an examination.

Direction A: knot complex to curves. The LOT Chapter 11 construction sends CFK^- to a bordered type D module. HRW23, Theorem 1.5 and the classification in §4 convert that module to decorated immersed curves. HRW22, Theorem 11 packages this dependence. Its §4.2 supplies the **effective geometric algorithm** (Propositions 46 and 48 in the simultaneous-basis case and the crossover-arrow method for arbitrary bases).

Direction B: curves to filling-core knot Floer homology. HRW22, Theorem 52 identifies gC^- and $g\widehat{C}$ with the LOT knot Floer box product, using the explicit type A operations (6.4). Its Remark 53 replaces the doubled-basepoint closed meridian by a proper slope arc. Introductory Proposition 12 and §4.4 identify its relative Spin-c summands; Proposition 56 matches the Alexander grading with height. HRW23, Example 6.8, checks the trefoil meridian calculation from the graded-curve construction.

What is outside scope: These two HRW papers do not provide a general, unqualified identification of the naive full $C^-(\mathscr{C}(M), \mu)$ with CFK^-. This is *explicitly* excluded by HRW22, §4.3. The cylindrical formulation and LOT are supporting reference systems, not replacements for the two HRW arguments.

11. Serious exercises

EXERCISE: Exercise 1 — Two basepoints and the filtration (15 minutes) A disk $B : x \to y$ has $(n_w, n_z) = (2, 1)$ and $A(x) = 3$. Determine $A(y)$, $A(U^2 y)$, and whether the term survives in $gCFK^-$ and $\widehat{gCFK}$. Repeat for $(n_w, n_z) = (1, 0)$. **Hint:** Use $A(x) - A(y) = n_z - n_w$ and then subtract n_w.

EXERCISE: Exercise 2 — All three trefoil homologies (25 minutes) For $\partial b = Ua + c$ with grades (4.1), compute $H_*(CFK^-)$, $H_*(gCFK^-)$, $H_*(\widehat{CFK})$, and $\widehat{HFK}$; distinguish the latter two. Recover the Alexander polynomial from the Maslov-graded Euler characteristic. **Hint:** Keep track of whether the filtration quotient was taken before or after setting $U = 0$.

EXERCISE: Exercise 3 — Construct the trefoil cylinder (20 minutes) Use Proposition 48 to draw the segments at heights $1, 0, -1$, attach the vertical and horizontal arcs on the correct sides, and connect the unmatched endpoints through the cylindrical identification. State which generator represents the distinguished vertical-homology class and read τ, ϵ. **Hint:** Check HRW22, Figure 3 and the discussion following Proposition 48.

EXERCISE: Exercise 4 — The algebra of a basis change (20 minutes) Let $D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P = I + E_{12}$ over $\mathbb{F}_2$. Compute PDP^{-1} and verify $D'^2 = PD^2P^{-1}$. Explain why a transformed graph need not remain visibly valence two. **Hint:** Retain the quadratic term $E_{12}DE_{12}$.

EXERCISE: Exercise 5 — Complete the proof of Theorem 52 (30 minutes) Write the type A operation $m_{3+i}(x_0, \rho_3, \rho_{23}^i, \rho_2)$ for $i = 0, 1, 2$. Identify the number of intermediate ι_1 generators and the corresponding w-multiplicity. Explain why only bigons with $n_z = 0$ enter the proven complex. **Hint:** Read HRW22, Theorem 52 and LOT18, Lemma 11.20.

EXERCISE: Exercise 6 — The wrong converse (15 minutes) Critically assess the assertion: "The curve invariant recovers all terms of CFK^- by counting every two-basepoint bigon." Identify the exact warning of HRW22 and give a logically correct replacement. **Hint:** Explain the difference between C^- and gC^-.

EXERCISE: Exercise 7 — Spin-c and height (20 minutes) Prove $A(\mathfrak{s} + a) - A(\mathfrak{s}) = a \cdot \lambda$ using the affine action on the first Chern class. Show the same formula for lifted-arc heights and explain the normalization ambiguity. **Hint:** HRW22, Proposition 56.

EXERCISE: Exercise 8 — Another filling core (20 minutes) For a solid torus whose meridian is $\alpha = (1,0)$, consider primitive filling slope $\mu = (3,7)$. Compute the minimal geometric intersection number and the rank of hat knot Floer homology of the filling core by the curve pairing. Explain which assertion is geometric and which relies on the HRW theorem. **Hint:** Minimal straight representatives have no bigons.

12. Solutions and hints

Exercise 1. If $(n_w, n_z) = (2,1)$, then $3 - A(y) = 1 - 2 = -1$, so $A(y) = 4$ and $A(U^2y) = 2$: the filtration drops by one and the term is absent from both associated-graded complexes. If $(1,0)$, then $A(y) = 4$ and $A(Uy) = 3$: it preserves Alexander filtration and survives in $gCFK^-$, but not in $g\widehat{CFK}$ because setting $U = 0$ kills the term.

Exercise 2. Total CFK^- has $H \cong \mathbb{F}[U]\langle[a]\rangle$ with relation $[c] = U[a]$. The Alexander associated graded has $\partial_{\mathrm{gr}} b = Ua$, so $HFK^- \cong \mathbb{F}[U]\langle[c]\rangle \oplus (\mathbb{F}[U]/(U))\langle[a]\rangle$. In $\widehat{CFK}$ the differential is $b \mapsto c$, giving rank-one $\widehat{HF}(S^3)$. In $g\widehat{CFK}$ the differential is zero, giving three rank-one Alexander summands at $1, 0, -1$. Their Maslov signs yield $t - 1 + t^{-1}$.

Exercise 3. At the left: b joins c; at the right: b joins a. The unmatched left end is a, and the unmatched right end is c. After the unstable connection the curve goes once around the cylinder. The unmatched vertical-homology generator is a at Alexander level 1; thus $\tau = 1, \epsilon = 1$, and the nonvertical slope in the prescribed coordinate system is 1.

Exercise 4. Since $P^{-1} = P$, direct multiplication gives $D' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Over $\mathbb{F}_2$, the upper-left 2×2 block squares to zero because every entry is the sum $1 + 1 = 0$. Hence $(D')^2 = 0 = PD^2P^{-1}$. The new basis presents the same chain complex but has four visible nonzero matrix entries rather than one.

Exercise 5. For $i = 0, 1, 2$ the operations are $m_3(x_0, \rho_3, \rho_2) = Ux_0$, $m_4(x_0, \rho_3, \rho_{23}, \rho_2) = U^2x_0$, and $m_5(x_0, \rho_3, \rho_{23}, \rho_{23}, \rho_2) = U^3x_0$. The intermediate ι_1 generators number $i + 1$, matching w-multiplicity $i + 1$. Only $n_z = 0$ matches the Alexander associated-graded LOT box product.

Exercise 6. False in general. HRW22, §4.3 explicitly warns that the unrestricted C^- may have $d^2 \neq 0$. The correct theorem is Theorem 52, which recovers $gCFK^-$ and $g\widehat{CFK}$ from the $n_z = 0$ bigon counts.

Exercise 7. $c_1(\mathfrak{s} + a) = c_1(\mathfrak{s}) + 2\,\mathrm{PD}(a)$. Pair with $[\Sigma]$ and divide by two to obtain $\langle \mathrm{PD}(a), [\Sigma]\rangle = a \cdot \lambda$. Deck translation changes the test arc's height by the same intersection. Thus the two functions agree up to an overall constant, fixed by normalizing one test lift.

Exercise 8. The absolute determinant is $|\det \left(\begin{smallmatrix} 1 & 3 \\ 0 & 7 \end{smallmatrix}\right)| = 7$. In the minimal straight-line configuration, the seven intersection generators survive. The geometric no-bigon check gives differential zero; HRW22, Remark 53 / Proposition 12 identifies their homology with $\widehat{HFK}$ of the filling core, so the total rank is seven.

13. Oral qualifying questions with answers

QUESTION: Q1. State the two directions of the knot-complement/immersed-curve correspondence. Model answer. HRW22, Theorem 11 says $CFK^-(K)$ determines the *hat bordered* curves of $S^3 \smallsetminus \nu K$ using LOT and HRW23. Conversely, for any filling core K_μ, Proposition 12 and Theorem 52 recover $\widehat{HFK}$, and also the associated-graded HFK^-, by pairing the curves with a slope-μ test object. This is not a theorem reconstructing all filtered differentials of CFK^- from the bare curves.

QUESTION: Q2. Why are horizontal and vertical simplified bases separate notions? Model answer. Vertical simplification concerns leading terms modulo U ($n_w = 0$). Horizontal simplification concerns terms preserving Alexander filtration ($n_z = 0$). They can require different filtered bases. A common basis gives a valence-two graph (Proposition 46); otherwise, the basis change is expressed by crossover arrows and must be simplified using HRW23.

QUESTION: Q3. Explain precisely the four steps of Proposition 48. Model answer. Place one horizontal segment for each CFK^- basis generator at its Alexander height; connect vertical arrows on the left; connect horizontal arrows on the right; connect the unique free ends through the cylinder to realize the unstable chain. The left/right conventions follow from the two different type D chain labels.

QUESTION: Q4. What is the essential warning in HRW22, §4.3? Model answer. The full naive two-basepoint C^- of curves need not have square-zero differential. Only its $n_z = 0$ associated-graded versions are proved to be chain complexes and to agree with $gCFK^-$ and $\widehat{gCFK}$; this is Theorem 52.

QUESTION: Q5. Why is the noncompact slope arc different from the closed slope curve? Model answer. A closed slope curve is used in the ordinary Dehn-filling pairing for $\widehat{HF}(M(\mu))$. The arc L_μ has ends at the removed puncture; its intersection Floer homology with $\mathscr{C}(M)$ computes the *hat knot Floer homology of the filling core*. The extra puncture and its placement distinguish the two basepoints of the knot.

QUESTION: Q6. Demonstrate the difference between $\widehat{HF}(S^3)$ and $\widehat{HFK}(T(2,3))$ on a single model. Model answer. For $\partial b = Ua + c$, the ordinary hat quotient has $\partial b = c$, with rank-one homology. The **Alexander associated-graded** hat quotient has zero differential and three homology generators at Alexander gradings $-1, 0, 1$. The total minus homology is one free $\mathbb{F}[U]$ tower, whereas the associated-graded minus homology additionally has a U-torsion summand.

QUESTION: Q7. How does a relative Spin-c class appear as an Alexander grading in the curve model? Model answer. Fix the graded cover and a lift $L_{\mu,\tilde{s}}$. Deck translation by $a \in H_M$ changes both its height and the Alexander grading by $a \cdot \lambda$. Normalizing at one lift yields equality of height and Alexander degree (HRW22, Proposition 56; HRW23, Example 6.8).

14. Reading schedule and proof checkpoints

Time	Close reading / task	Deliverable
40 min	HRW22 §4.1 and intro Theorems 11–12	Keep CFK^-, $gCFK^-$, $\widehat{CFK}$, and $\widehat{gCFK}$ distinct
65 min	HRW22 §4.2, Propositions 46, 48; Figures 36–40	Reconstruct the two simplified-basis constructions and the cylindrical recipe
50 min	Trefoil model (4.1)–(4.6); HRW23 Example 6.8	Compute all homologies and sketch the actual immersed curve
55 min	HRW22 §4.3, Theorem 52 and Remark 53	Prove the box-product/bigon dictionary; identify the dangerous C^- warning
35 min	HRW22 §4.4, Proposition 56	Recover Alexander grading from lift heights
35 min	Exercises 1, 2, 5, 6 and oral rehearsal	Give precise board-level proofs and caveats

Total: 4 hours 50 minutes.

Proof checkpoints

- ☐ Derive $A(x) - A(y) = n_z - n_w$ and the filtration drop of $U^{n_w}y$.
- ☐ Differentiate CFK^-, $gCFK^-$, $\widehat{CFK}$, and $\widehat{gCFK}$ without a diagram.
- ☐ Explain why a common simplified basis yields a valence-two type D graph.
- ☐ Reproduce HRW22, Proposition 48, including left/right attachment and the unstable chain.
- ☐ Compute the three-generator trefoil complex in all four flavors and recover $\Delta_K(t)$.
- ☐ Explain how a nontrivial filtered basis change creates crossover arrows.
- ☐ State Theorem 52 and write the type A operations m_{3+i} precisely.
- ☐ State the exact d^2 warning for naive C^-.
- ☐ Distinguish a closed filling-slope curve from the proper test arc L_μ.
- ☐ Derive the Alexander-height formula from c_1 and deck translations.

15. Verified references and next lecture

[HRW22] J. Hanselman, J. Rasmussen, L. Watson, *Heegaard Floer homology for manifolds with torus boundary: properties and examples*, *Proceedings of the London Mathematical Society* **125** (2022), 879–967. Theorem 11, Proposition 12 (Introduction); §4.1 (background); §4.2, Propositions 46 and 48, Remark 47, Examples 49–50, Figures 36–40; §4.3, Theorem 52, Remark 53, Figures 41–42; §4.4, Proposition 56 and Figure 43. Published article; arXiv:1810.10355v1 (different numbering). *The preprint has Proposition 45/47, Theorem 51, and Proposition 55 where the published version has Proposition 46/48, Theorem 52, and Proposition 56.*

[HRW23] J. Hanselman, J. Rasmussen, L. Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, *Geometry & Topology* **27** (2023), 823–924. Theorems 1.1 and 1.5; §3.7 (curve reduction); §4 (pairing/classification); §6.2, Example 6.8 and Figure 47 (trefoil Alexander levels). arXiv:1604.03466v3.

[LOT18, supporting] R. Lipshitz, P. Ozsváth, D. Thurston, *Bordered Heegaard Floer Homology*, *Memoirs AMS* **254** (2018), no. 1216, Chapter 11, especially Theorem 11.19 and Lemma 11.20 as cited by HRW22. AMS DOI.

Next (Day 11): HRW22, §5: Alexander polynomial, Turaev torsion, and the Thurston norm. The key transition is from **intersections with test arcs** to **oriented intersection numbers with rays from lifts of the puncture**; we will distinguish geometric intersection counts from algebraic intersection numbers.