Day 12 - L-space Intervals from Torsion and Tangent Slopes

Course date: Tuesday, November 3, 2026. **Study time:** 4 h 50 min. **Level:** geometry/topology PhD qualifying exam. **Coefficients:** $\mathbb{F}_2$ for Floer homology; integer coefficients for Turaev torsion.

Companion PDF: HRW_Day12.pdf
Concept links: Immersed Curves · Turaev Torsion · L-space Intervals · Floer Simple Manifolds · Dehn Filling · Pegboard Diagrams · Bordered Pairing Theorem · Local Systems

> **Warning** Source and scope conventions All HRW22 theorem numbers here refer to the **published** *Proceedings of the London Mathematical Society* paper (2022), not its 2018 arXiv first version, where Section 5.3 is numbered one lower (Theorem 68 instead of 69). HRW23 means the **three-author** *Bordered Floer homology for manifolds with torus boundary via immersed curves*, arXiv:1604.03466v3 (2023), not a different two-author *A calculus for bordered Floer homology*. Statements marked **Derived** are proved in this lecture; diagrams/analysis not reproduced are marked **proof sketch**. Supporting sources are subordinate to these two HRW papers.

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1. Goals and exact theorem map

Yesterday we identified the Turaev torsion of a torus-boundary manifold with signed intersection numbers between the Spin^c-refined HRW curve and downward rays. Today we extract **which filling slopes give L-spaces**. The key novelty is that an L-space interval is controlled not by unsigned intersection numbers alone, but by *tangent directions of tight curves*; for Floer-simple manifolds these in turn are forced by the pattern of 0's and 1's in the Turaev torsion.

The central chain of results, in the order needed for a proof, is

HRW23 Theorem 1.2: filling as curve pairing
$\Downarrow$
HRW23 Theorem 7.17: L-space interior versus singular slopes
$\Downarrow$
HRW22 Lemma 66: a Floer-simple lift is a graph
$\Downarrow$
HRW22 Proposition 62: torsion = signed ray intersections
$\Downarrow$
HRW22 Proposition 67 + Lemma 68
$\Downarrow$
HRW22 Theorem 69: the interval from $D^\tau(M)$.

The 2023 paper proves a **general tangent-slope criterion** and an **L-space gluing theorem**. The 2022 paper proves that in the Floer-simple case the tangent-slope interval can be reconstructed *purely from torsion*. The two are complementary, not interchangeable. Our most important conceptual task is to explain exactly where the Floer-simple and non-solid-torus-like hypotheses enter.

Targets for an oral exam: derive the graph lemma, prove the black-white torsion coloring, reconstruct the two inclusions in Proposition 67, explain the projectivized preimage in Lemma 68, and state Theorem 69 without changing an interior into a closed interval.

2. Slopes, L-spaces, and source conventions

Let M be a compact oriented connected three-manifold with $\partial M \cong T^2$ and a distinguished point z. Write $T_M = \partial M \setminus \{z\}$. Let $\mathcal{C}(M) = \widehat{HF}(M)$ denote the **decorated immersed-curve invariant** (HRW23, Theorem 1.1); it is not the closed-manifold vector space $\widehat{HF}(Y)$.

A *slope* is an unoriented primitive homology class in $H_1(\partial M; \mathbb{Z})$, or equivalently a point of

$$\mathrm{Sl}(\partial M) \cong \mathbb{Q}P^1.$$

Fix a basis (μ, λ) only when needed. A primitive class $p\mu + q\lambda$, $\gcd(p, q) = 1$, is often recorded by the surgery coordinate p/q; the vectors (p, q) and $(-p, -q)$ describe the same slope. The value ∞ means $q = 0$. In the geometry of HRW22, Section 5.3, one instead uses coordinates adapted to a *chosen interior L-space slope* α; the resulting Euclidean slope of a tangent line is generally **not** the numerical surgery coefficient p/q. We always state which coordinate is in use.

A rational homology sphere Y is an **L-space** if

$$\dim_{\mathbb{F}} \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|.$$

Put

$$\mathcal{L}(M) = \{\beta \in \mathrm{Sl}(\partial M) : M(\beta) \text{ is an L-space}\}.$$

The interior $\mathcal{L}^\circ(M)$ is taken in the rational slope circle $\mathbb{Q}P^1$ (with its order topology inherited from $\mathbb{R}P^1$), not in a chosen real affine chart. HRW22, Section 5.3 calls M **Floer simple** if $\mathcal{L}^\circ(M) \neq \emptyset$; a standard equivalent description in the situation at hand is that M has more than one L-space filling. A *solid-torus-like* manifold is one whose tight curve components do not have genuine pegboard corners in the relevant sense; it is a separate exceptional class in the statements below.

> **Warning** Four frequent mistakes (i) The meridian of the **knot exterior** need not be the curve representing the **longitude of a solid torus** in the same marking. (ii) The slope ∞ is a single point on the slope circle, not an unbounded direction that can be ignored when describing connected intervals. (iii) Interior $\mathcal{L}^\circ$ is not automatically the entire L-space set $\mathcal{L}$; endpoints require separate analysis. (iv) The theorem does **not** calculate an interval from torsion alone in the absence of a certified $\alpha \in \mathcal{L}^\circ(M)$.

2.1 The central source theorem in the main paper

> **Theorem** HRW23, Theorem 7.17 (exact mathematical content) For a manifold with torus boundary,
>
> $$\mathcal{L}^\circ(M) = \mathbb{Q}P^1 \setminus S^{\mathrm{sing}}_{\mathbb{Q}}(M),$$
>
> where $S^{\mathrm{sing}}_{\mathbb{Q}}(M)$ consists of rational *essential tangent slopes* of the HRW curve collection, defined using minimal tight representatives. This is the 2023 input to the torsion theorem.

A tangent slope obstructs stable minimal intersection with the straight test circle of that slope. A nonsingular slope, by contrast, can be realized so that the pairing with a straight slope curve is minimal within every relevant Spin^c summand. This is how the geometry detects L-space fillings. The full direction "non-L-space implies singularity'' requires the bigon/corner and local-system arguments in HRW23, Section 7.5; we mark those below as an external input rather than pretending that simple transversality alone proves it.

3. Why Floer simplicity forces a graph

Fix $\alpha \in \mathcal{L}^\circ(M)$. Let l denote the **primitive rational longitude**, distinguished from the possibly nonprimitive homological longitude λ. Following HRW22, Section 5.3, identify the real boundary homology plane with $\mathbb{R}^2$ using the map

$$\beta \longmapsto (\beta \cdot \alpha,\ \beta \cdot l). \tag{3.1}$$

The slope α becomes the vertical direction. Lift a tight representative of $\mathcal{C}(M, \mathfrak{s})$ to the punctured plane $H_1(\partial M; \mathbb{R}) \setminus H_1(\partial M; \mathbb{Z})$ and write $\tilde{\gamma}_{\mathfrak{s}}$ for the lift. Here we are using the grading-cover refinement of HRW23, Definition 1.6 and Theorem 1.7, not an arbitrary drawing of an immersed loop.

> **Theorem** HRW22, Lemma 66 For every $\mathfrak{s} \in \mathrm{Spin}^c(M)$, the tightened lifted pegboard curve $\tilde{\gamma}_{\mathfrak{s}}$ is the graph of a function
>
> $$y = f_{\mathfrak{s}}(x)$$
>
> in the α-adapted coordinates (3.1).

Proof Source proof, expanded Choose c so the vertical test line $L_c = \{x = c\}$ avoids all pegs, and tighten it together with the curve. Since $M(\alpha)$ is an L-space, the relevant straight-line pairing has the minimal possible rank. In the Spin^c summand in question the line meets the lifted curve **once** in minimal position (HRW22, Lemma 66). Therefore projection of the lifted curve onto the x-axis has precisely one point in the inverse image of each generic $x = c$. This makes the lift a graph except possibly for vertical segments linking pegs. If such a segment existed, α would be a tangent/singular slope. But $\alpha \in \mathcal{L}^\circ(M)$, which is disjoint from $S^{\mathrm{sing}}(M)$ by HRW23, Theorem 7.17. Therefore no vertical segment occurs; the entire tightened lift is a graph. □

The graph is embedded. Consequently $\mathbb{R}^2 \setminus \tilde{\gamma}_{\mathfrak{s}}$ has exactly two components: a region toward $\beta \cdot l \ll 0$, and a region toward $\beta \cdot l \gg 0$. The graph property is much stronger than merely stating that the curve has trivial local system; it imposes a global order on the lifted strands.

Warning Which step really uses interior? The fact that a vertical test line has the minimal number of intersections follows from an L-space *filling*. The exclusion of vertical pegboard segments requires α to be an **interior** L-space slope. At a boundary L-space slope, the curve can develop tangent segments; the conclusion of Lemma 66 is not justified by the rank condition alone.

3.1 A derived algebraic rank check

Let L_α be the rank-one slope curve representing the filling solid torus. If the immersed-curve model is transverse and every differential bigon is absent in minimal position, then

$$\dim HF(\mathcal{C}(M), L_\alpha) = \#(\mathcal{C}(M) \cap L_\alpha)$$

in the trivial-local-system case, with component and lift multiplicities included. However, if a pair is not in minimal position, the raw intersection count exceeds the Floer rank by twice the rank of a differential:

$$\dim H(C) = \dim C - 2\,\mathrm{rank}(d) \tag{3.2}$$

for any finite-dimensional $\mathbb{Z}/2$-graded chain complex C with differential $d : C \to C$. We prove (3.2) later as an elementary diagnostic; it explains why accidental extra intersections are not evidence of failure of the pairing theorem.

4. Torsion as a coloring of the grading cover

For each $\mathfrak{s} \in \mathrm{Spin}^c(M)$ the punctures in the lifted plane are indexed by relative Spin^c structures restricting to $\mathfrak{s}$, as reviewed on Days 8 and 11. HRW22, Proposition 62 identifies Turaev torsion with **signed intersection** with a ray from the puncture to the negative end:

$$\tau_M(\bar{\mathfrak{s}}) = \gamma_{\bar{\mathfrak{s}}} \cdot \mathcal{C}(M, \mathfrak{s}). \tag{4.1}$$

This is an equality of integers, not merely a parity or intersection-count identity. It depends on the homology orientation and the chosen affine normalization of the relative Spin^c torsor; see HRW22, Lemma 61 and Proposition 62.

In the graph situation of Lemma 66, the ray from a peg below the graph lies on the *negative* side, so its signed count is 0. The ray from a peg above the graph crosses the oriented graph once and has signed count 1. Thus:

$$\boxed{\tau_M(\bar{\mathfrak{s}}) \in \{0, 1\}, \qquad \begin{array}{l}\text{black peg} \iff \tau = 0,\\ \text{white peg} \iff \tau = 1.\end{array}} \tag{4.2}$$

Here **black** denotes the component containing the region with $\beta \cdot l \ll 0$; **white** denotes the other component, as in HRW22, after Lemma 66. These are colors of *the puncture lattice in a fixed Spin^c lift*, not arbitrary paint on the original torus.

Proof Why the values are exactly $0, 1$ The graph is an oriented proper embedded copy of $\mathbb{R}$ dividing the plane. A downward ray based below it can be chosen in the same complementary component; its signed intersection vanishes. A ray based above it must cross the graph, and the algebraic intersection is $+1$ with HRW's homology orientation. Extra pairs of crossings created by perturbation cancel algebraically. Equation (4.1) then gives (4.2). The source's orientation normalization is indispensable: reversing every curve would turn $+1$ into -1 rather than proving the same coloring. □

4.1 Contrast with an arbitrary knot exterior

For the right-handed trefoil $\Delta(t) = t^{-1} - 1 + t$. With the usual formal expansion,

$$T(t) = \frac{\Delta(t)}{t^{-1/2} - t^{1/2}} = t^{-1/2} + t^{3/2} + t^{5/2} + \cdots.$$

Hence the torsion values along one affine $\mathbb{Z}$-orbit are 0 sufficiently low, 1 at $-1/2$, 0 at $+1/2$, and 1 at $+3/2$ and above. These are all in $\{0, 1\}$, consistent with Floer simplicity. For the figure-eight knot $\Delta(t) = -t^{-1} + 3 - t$ the analogous sequence includes -1 and 2; its knot exterior is not Floer simple. This is a useful **necessary** diagnostic, but the converse "torsion takes only $0, 1$ implies Floer simple'' does not follow without further assumptions.

4.2 A rigorous toy graphic (derived, not an HRW invariant)

To visualize what can be extracted from color data, consider the smooth graph

$$f(x) = \tfrac{1}{2}x + \tfrac{1}{16}\sin(2\pi x) \quad (x \in \mathbb{R}).$$

It satisfies $f(x+1) = f(x) + 1/2$. Its tangent slopes are

$$f'(x) = \tfrac{1}{2} + \tfrac{\pi}{8}\cos(2\pi x),$$

so the complete tangent interval in ordinary Cartesian coordinates is

$$\boxed{[\tfrac{1}{2} - \tfrac{\pi}{8},\ \tfrac{1}{2} + \tfrac{\pi}{8}].} \tag{4.3}$$

This graph illustrates the **calculus** behind essential tangent slopes. It is *not* claimed to come from a tight pegboard diagram or any three-manifold (a tight pegboard has rationally directed edges, unlike this smooth toy). Therefore (4.3) is **not** an L-space interval. In particular, one cannot insert the numbers in (4.3) into a surgery basis without applying the correct α-adapted coordinate transformation.

5. From black-white segments to tangent slopes

Let $p, q \in \mathbb{Z}^2$ be distinct pegs. Write $[p-q] \in \mathbb{R}P^1$ for the direction of their joining line. In the chosen α-adapted plane, define

$$X_{\mathfrak{s}} = \Big\{[p-q] :\ p \text{ black},\ q \text{ white},\ l \cdot (p-q) \geq 0\Big\}. \tag{5.1}$$

The sign condition is **not optional**. It selects the black/white pairs in the oriented order used by HRW. It should not be replaced with "take every black-white difference'' or with an inequality on a numerical surgery coefficient unless the basis and intersection convention have been specified.

Let $S^{\mathrm{sing}}(M, \mathfrak{s})$ denote the interval of *essential tangent slopes* for the tightened lifted curve; this is an interval in the complement of the chosen α, not an arbitrary set of slope coefficients. The precise tangent notion is developed in HRW23, Section 7.5.

> **Theorem** HRW22, Proposition 67 (published version) If M is Floer simple and **not solid-torus-like**, and $\alpha \in \mathcal{L}(M)$ is the chosen reference L-space slope, then
>
> $$S^{\mathrm{sing}}(M, \mathfrak{s}) = \text{the smallest interval in } \mathbb{R}P^1 \setminus \{\alpha\} \text{ containing } X_{\mathfrak{s}}. \tag{5.2}$$

The important geometric mechanism has two directions, and they should be reconstructed separately.

5.1 Every black-white direction is singular

Take p black and q white with $l \cdot (p-q) > 0$. (The equality case is treated by the limiting/endpoint convention in the source.) Follow the ray that begins at p, goes through q, and continues beyond. Because p is below the graph and q is above it, the segment $[p, q]$ meets the graph at some point x_1. The oriented slope condition means that the continuation of this ray eventually returns to the black side, so it meets the graph a second time x_2 beyond q. On the straight line through p, q, write the graph as $y = f(x)$ and the line as $y = mx + b$ (away from the vertical reference direction). At the two intersections,

$$f(x_1) - (mx_1 + b) = f(x_2) - (mx_2 + b) = 0.$$

By Rolle's theorem, some $x_* \in (x_1, x_2)$ satisfies

$$f'(x_*) - m = 0.$$

Hence the joining direction $[p-q]$ is a tangent slope. With a pegboard graph the differentiability step is replaced by the extended mean-value theorem, and a corner or segment of slope m is allowed. Therefore

$$\boxed{X_{\mathfrak{s}} \subseteq S^{\mathrm{sing}}(M, \mathfrak{s}).} \tag{5.3}$$

Warning Where this argument can fail Merely knowing that a line joining opposite-color points intersects a graph **once** does not imply tangency. We need the second intersection. HRW's oriented condition and the asymptotic geometry of the graph provide that extra intersection. Dropping the sign condition on $l \cdot (p - q)$ invalidates the argument.

5.2 Every extremal tangent slope comes from black-white pegs

Because M is **not** solid-torus-like, the essential-tangent set has endpoints attained on genuine edges/corners of the tightened pegboard. In the coordinates where α is vertical, an endpoint is an extreme slope of f, namely an extreme value of its piecewise-linear derivative f'. At a maximum-slope edge, concavity changes from upward to downward. The adjacent pegs on the two sides have opposite colors: the edge is supported by a white peg at one end and a black peg at the other. Its direction is therefore a member of $X_{\mathfrak{s}}$. The minimum-slope edge is treated similarly.

It follows that both endpoints of the closed singular interval lie in $X_{\mathfrak{s}}$. Together with (5.3), this proves (5.2).

Proof Proof status for Proposition 67 The first inclusion has been proved above up to the straightforward endpoint/generalized mean-value cases. The endpoint argument is a faithful **diagrammatic proof sketch** of HRW22, Proposition 67: its pegboard corner types and the assertion that the extrema are supported by opposite-colored pegs are checked in the original source. We do not claim to have reproved the classification of all pegboard corner configurations here.

5.3 A useful derived calculus lemma

Let f be continuously differentiable on $[a, b]$ and let m be the slope of the secant through $(a, f(a))$ and $(b, f(b))$. Then there exists $x_* \in (a, b)$ with $f'(x_*) = m$.

Proof. Apply Rolle's theorem to $g(x) = f(x) - mx$. Because $g(a) = g(b)$, some $g'(x_*) = 0$. This gives $f'(x_*) = m$. □

This elementary lemma is the analytic core of the black-white slope argument, but it does not replace the global graph and pegboard hypotheses.

6. From peg differences to the torsion difference set

The next step is algebraic. Relative Spin^c structures form an affine torsor over $H_1(M; \mathbb{Z})$. Let τ_M be the normalized relative Turaev torsion. Let $j_* : H_1(\partial M; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ be inclusion and l the primitive rational longitude. Following the published HRW22, Section 5.3, set

$$D^\tau(M) = \{\bar{\mathfrak{s}}_0 - \bar{\mathfrak{s}}_1 : \ \tau_M(\bar{\mathfrak{s}}_i) = i \ (i = 0, 1), \ l \cdot (\bar{\mathfrak{s}}_0 - \bar{\mathfrak{s}}_1) \geq 0\} \subset H_1(M). \tag{6.1}$$

Here the last pairing is the longitude intersection/evaluation defined on relative Spin^c differences. The equation $\tau(\bar{\mathfrak{s}}_0) = 0$ and $\tau(\bar{\mathfrak{s}}_1) = 1$ is asymmetric on purpose. **Do not reverse the subscripts.**

Now take the complete preimage

$$j_*^{-1}(D^\tau(M)) \subset H_1(\partial M; \mathbb{Z})$$

and projectivize its **nonzero** elements to slope classes. Denote the result by

$$\mathbb{P}(j_*^{-1}(D^\tau(M)) \setminus \{0\}). \tag{6.2}$$

Theorem HRW22, Lemma 68

$$\boxed{\mathbb{P}(j_*^{-1}(D^\tau(M))) = \bigcup_{\mathfrak{s} \in \mathrm{Spin}^c(M)} X_{\mathfrak{s}}.} \tag{6.3}$$

The projectivization implicitly discards zero vectors.

Proof Expanded source proof Two relative Spin^c structures $\bar{\mathfrak{s}}_0, \bar{\mathfrak{s}}_1$ have difference in $\operatorname{im} j_*$ **if and only if** their images in the absolute Spin^c set agree. Indeed, passing from the relative to the absolute torsor quotients the difference lattice by $\operatorname{im} j_*$. Choose this common absolute $\mathfrak{s}$. Lift the corresponding punctures to lattice points p_0, p_1 in the universal punctured plane. Every vector in $j_*^{-1}(\bar{\mathfrak{s}}_0 - \bar{\mathfrak{s}}_1)$ is realized by an appropriate lattice difference $p_0 - p_1$, including all deck translates. By HRW22, Proposition 62 and Lemma 66, the conditions $\tau(\bar{\mathfrak{s}}_0) = 0$, $\tau(\bar{\mathfrak{s}}_1) = 1$ are precisely `$p_0$ black''` `and`p_1 white''. The order inequality becomes $l \cdot (p_0 - p_1) \geq 0$. Consequently the projectivized differences are exactly the directions in $X_{\mathfrak{s}}$. Taking the union over $\mathfrak{s}$ gives (6.3). □

This is a key oral-exam point: a difference in $H_1(M)$ is not itself a boundary slope. It becomes slope data only after taking **the whole boundary preimage** and then projectivizing. If j_* has a nontrivial kernel, replacing the preimage by one arbitrarily chosen lift loses information. If the longitude is nonprimitive in the integral lattice, the torsion in $H_1(M)$ must also be retained.

6.1 An explicit lattice-preimage computation (derived)

Take an abstract lattice map

$$j_* : \mathbb{Z}\mu \oplus \mathbb{Z}\lambda \to \mathbb{Z} \oplus \mathbb{Z}/3, \qquad j_*(a\mu + b\lambda) = (a, [b]_3).$$

Then $\ker j_* = \langle 3\lambda \rangle$, and

$$j_*^{-1}(\{(2, [1]_3)\}) = \{2\mu + (1 + 3k)\lambda : k \in \mathbb{Z}\}. \tag{6.4}$$

For a **slope**, retain only primitive vectors. For instance, $2\mu + \lambda$ is primitive, whereas $2\mu + 4\lambda$ is not; the latter has the same rational line as $\mu + 2\lambda$, but is *not* in the fiber as a primitive boundary homology class. In applying (6.2), one projectivizes nonzero integer vectors; a multiple that is not primitive still has a direction, but it must not be confused with a distinct primitive integral lift. The entire set of directions is

$$\{[2 : 1 + 3k] : k \in \mathbb{Z}\}.$$

This is a **model homomorphism**, not a claimed torsion difference set of a particular three-manifold. It demonstrates both the kernel issue and the necessity of specifying slope coordinates.

7. The exact torsion-to-interval theorem

Theorem HRW22, Theorem 69 (published version) Suppose M is **Floer simple**, **not solid-torus-like**, and $\alpha \in \mathcal{L}^\circ(M)$ is known. Then $\mathcal{L}^\circ(M)$ is the **largest interval of** $\mathrm{Sl}(\partial M)$ that contains α and contains no slope in

$$\boxed{\mathbb{P}(j_*^{-1}(D^\tau(M)))}. \tag{7.1}$$

Proof Assembly of the published proof By HRW23, Theorem 7.17, $\mathcal{L}^\circ(M)$ is exactly the complement of the rational essential-tangent slopes. For each $\mathfrak{s}$, HRW22, Proposition 67 identifies the **smallest singular interval avoiding** α with the interval hull of $X_{\mathfrak{s}}$. Hence the connected component of the complement of the union of these singular intervals that contains α is the largest interval disjoint from $\bigcup_{\mathfrak{s}} X_{\mathfrak{s}}$. Lemma 68 identifies that union with the projectivized boundary preimage of $D^\tau(M)$. Therefore the largest interval containing α and avoiding the torsion difference directions is $\mathcal{L}^\circ(M)$. □

The proof uses *two* ingredients that are sometimes omitted in informal summaries. First, the total tangent set is an interval (or union with a controlled interval hull) **in the complement of the chosen reference direction**. Second, the reference slope α selects the correct component of the slope circle after the forbidden slopes are removed. Without α, a torsion calculation does not specify which complementary component is the L-space interval.

Warning The endpoints are not settled by Theorem 69 Theorem 69 reconstructs $\mathcal{L}^\circ(M)$. It does not say that every boundary direction of its closure is an L-space filling. To decide whether an endpoint itself belongs to $\mathcal{L}(M)$, one must use the full tangent-line geometry and/or direct Floer calculations. This matters for integer surgery thresholds such as $2g - 1$ for positive L-space knots.

7.1 A safe algorithm for an actual computation

Assume all required source hypotheses are verified, including a certified $\alpha \in \mathcal{L}^\circ(M)$. (1) Fix the homology orientation and normalized relative Spin^c torsor. (2) Compute τ_M with its integer coefficients and torsion classes. (3) Form all ordered 0-to-1 differences satisfying the l-ordering in (6.1). (4) For each difference, take *all* boundary lattice preimages under j_*. (5) Projectivize nonzero preimages to the actual slope circle. (6) Locate the largest

connected interval containing α and avoiding these directions. (7) Check its endpoints separately. This is an algorithmic **logical recipe**; it is not a claim that every infinite torsion-support computation terminates without an appropriate finite representation.

8. Direct geometric computations and checks

The point of the next computations is to practice the objects that *really* appear in HRW's proofs: slope classes, determinant intersection counts, tangent intervals, and total homology ranks. We distinguish fully verified algebraic exercises from actual three-manifold classifications.

8.1 Primitive slope arithmetic

Fix an oriented basis (μ, λ) and define $\mu \cdot \lambda = +1$ for this computation. A primitive slope p/q is represented by $p\mu + q\lambda$. For (p, q) and (r, s),

$$\boxed{i_{\min}((p,q),(r,s)) = |ps - qr|.} \tag{8.1}$$

Derived proof. Lift the two primitive embedded circles to lines in $\mathbb{R}^2$, each invariant under translation by its integral direction. The image of the intersection lattice has index $|ps - qr|$ in $\mathbb{Z}^2$; one intersection occurs per coset after quotienting by the torus deck group. Straight representatives have no contractible bigons, so this is already the minimal geometric intersection number. □

Example: $(p, q) = (2, 3)$ and $(r, s) = (1, 4)$ give

$$i_{\min} = |2 \cdot 4 - 3 \cdot 1| = 5.$$

If one curve is the meridian circle of a solid torus and the other is the filling slope, the corresponding lens-space filling has first homology order 5 (provided the gluing produces a rational homology sphere), and the curve pairing has five generators and zero differential. Thus $\dim \widehat{HF} = 5$, consistent with the L-space condition. **This is a closed-curve filling calculation, not the noncompact test-arc calculation of knot Floer homology.**

8.2 Why extra intersection points can cancel

Consider a finite-dimensional chain complex over $\mathbb{F}_2$ with four generators a, b, c, d and the only differential $d(a) = b$, $d(c) = 0$, $d(d) = 0$, $d(b) = 0$. Then $d^2 = 0$, $\dim C = 4$, $\operatorname{rank} d = 1$, so

$$\dim H(C) = 4 - 2 = 2.$$

General derived proposition. If C is a finite-dimensional complex with $d^2 = 0$, then

$$\dim H(C) = \dim C - 2 \operatorname{rank} d. \tag{8.2}$$

Proof. Since $\operatorname{im} d \subseteq \ker d$,

$$\dim H = \dim \ker d - \dim \operatorname{im} d = (\dim C - \operatorname{rank} d) - \operatorname{rank} d.$$

This is exactly the algebraic effect of a canceling pair of bigon-related intersection generators. It is **not** a proof that every geometric bigon can be removed independently: admissibility and local systems still matter.

8.3 A rational slope interval can cross infinity

Identify the slope circle with $\mathbb{R} \cup \{\infty\}$. The arc

$$I = \{r \in \mathbb{R} : r > 2 \text{ or } r < -3\} \cup \{\infty\}.$$

This is the connected open projective arc from slope 2 through ∞ to slope -3. It includes the point ∞ as an *interior* point. Thus ∞ is not automatically an endpoint merely because an affine chart ends there.

A practical way to remove ambiguity is to represent directions by unit vectors modulo antipodal identification, $v \sim -v$. The slope circle is a circle, not the extended real line with its two tails kept separate. When checking Theorem 69, draw the **projective** circle and mark the forbidden directions before declaring the maximal interval.

8.4 A calculus test for tangent slopes

For the toy function in (4.3),

$$f'(x) = \frac{1}{2} + \frac{\pi}{8}\cos 2\pi x.$$

Its maximum and minimum are $1/2+\pi/8$ and $1/2-\pi/8$. A line with slope m is tangent if and only if m lies in this interval, because cosine assumes every value between -1 and 1. In particular $m = 1/2$ is tangent at $x = 1/4, 3/4$ modulo integers; $m = 2$ is not. Again: this is an *analytic analogue*, not an actual L-space computation.

8.5 The trefoil surgery threshold (comparison, not deduced from torsion coordinates here)

For the positive L-space knot $T(2,3)$, the standard Ozsv'ath–Szab'o L-space surgery theorem identifies the positive rational L-space surgery region as $p/q \geq 2g(K)-1 = 1$ (with the customary meridian–longitude surgery convention). Thus $+2$-surgery is an L-space and -2-surgery is not; meridional filling gives S^3. This example checks that one must take the correct **connected component of the projective slope circle**, and never infer a filling classification from the signless intersection determinant alone.

We do **not** claim to have reconstructed the trefoil's entire D^τ boundary-preimage set from the abbreviated half-integer torsion list; that reconstruction requires the explicit affine identification, the homology orientation, and the torsion-to-boundary map. In an oral exam, write these choices on the board *before* turning half-integer indices into p/q surgery slopes.

9. The main paper's tangent-slope theorem and gluing theorem

The main 2023 paper is indispensable here. The 2022 torsion result depends on HRW23's analysis of **which slope lines pair minimally** with the immersed-curve object. This comes from the pairing theorem and is developed in Sections 7.5–7.6.

> **Theorem** HRW23, Theorem 1.2 (pairing, relevant form) If $Y = M_0 \cup_h M_1$ is obtained by an orientation-reversing torus gluing, then
>
> $$\widehat{HF}(Y) \cong HF(\mathcal{C}(M_0),\, h(\mathcal{C}(M_1)))$$
>
> after the elliptic-involution convention is handled as in HRW22. The right side is immersed Lagrangian intersection Floer homology with local-system transport, not a naive count of curve crossings.

For a Dehn filling $M(\alpha)$, one of the pieces is a solid torus, whose curve invariant is a single embedded circle of the filling meridian slope. Thus $\widehat{HF}(M(\alpha))$ can be read as the Floer pairing of $\mathcal{C}(M)$ with a straight slope circle L_α. This is how an abstract condition on closed three-manifold rank turns into a geometric condition on tangent directions.

> **Theorem** HRW23, Theorem 1.14 (L-space gluing) Let $Y = M_0 \cup_h M_1$, where the two pieces are **boundary incompressible** and h is orientation reversing. Then Y is an L-space if and only if
>
> $$\boxed{\mathcal{L}^\circ(M_0) \cup h(\mathcal{L}^\circ(M_1)) = \mathbb{Q}P^1.} \tag{9.1}$$

This theorem is more general than the Floer-simple torsion calculation: one can determine the intervals from the curves without first computing Turaev torsion. HRW22, Theorem 69 supplies an efficient torsion-based way to obtain each interval **when Floer simplicity is known**. Consequently the theorems combine into a computational strategy for toroidal L-space detection:

$$\boxed{\text{torsion of each side} \Longrightarrow \mathcal{L}^\circ(M_i) \Longrightarrow \text{projectively compare the two arcs} \Longrightarrow \text{L-space gluing test.}}$$

The final implication is valid under the boundary-incompressibility assumptions of (9.1).

9.1 How the gluing theorem is proved: an explicitly incomplete proof sketch

HRW23, Section 7.6 proves the theorem by comparing the signed intersection data of curves with the minimal possible ranks in the relevant Spin^c sectors. If there exists a slope outside both interiors, it lies in the two singular-slope sets; the tight curves can then be positioned so that the pairing contains unnecessary pairs of intersections, obstructing the L-space rank condition. Conversely, if the two interior L-space intervals cover the slope circle, each relative direction is nonsingular on at least one side, so the curve pairing can be arranged without the excess intersections that would produce reduced Floer homology. The argument uses the geometric intersection estimates and the special properties of manifold-derived curves from Section 7.3. **Proof status: incomplete sketch**: the local bigon estimates and the boundary-incompressible reduction are not reproduced here. Use HRW23, Theorem 1.14 and Section 7.6 for the full proof.

9.2 A purely topological interval-cover test (derived)

To see how (9.1) works once the intervals are known, identify the slope circle abstractly with $\mathbb{R}/\mathbb{Z}$ using an orientation-preserving homeomorphism. Consider open arcs

$$I_0 = (0.1, 0.8) \pmod 1, \qquad I_1 = (0.6, 1.2) \pmod 1.$$

Their union is the whole circle: I_0 covers $(0.1, 0.8)$, while I_1 covers the remaining part from 0.6 past $1 \sim 0$ to 0.2. In contrast, $(0.1, 0.6) \cup (0.5, 0.9)$ misses the projective arc from 0.9 back to 0.1. These are **abstract interval exercises**, not claimed to arise from particular manifolds. They show why checking only that two intervals overlap is insufficient; one needs full coverage.

9.3 Local systems and multiplicity

Suppose an immersed curve carries a rank-r local system. At each transverse intersection with a rank-one test slope circle, the Floer chain group contributes an r-dimensional fiber. Consequently a rank-r component can force the Floer rank to exceed the minimal L-space rank; this motivates the trivial-local-system conclusions for Floer-simple manifolds in the HRW papers. We must not infer that *every abstract extendable type D module* has trivial local systems. The geometric classification of HRW23 permits arbitrary invertible monodromy; the L-space filling restrictions apply to a much smaller class of manifold-derived objects.

10. Solid-torus-like exceptions and diagnostic failures

The exclusion of solid-torus-like manifolds in Proposition 67 and Theorem 69 is mathematically substantial. A curve parallel to the rational longitude can be pushed into a narrow annulus avoiding pegs. It has no genuine corners, so its tangent/pegboard behavior is exceptional. It may admit L-space fillings of every slope other than its rational longitude, even though the ordinary black-white **corner argument** of Proposition 67 has no extremal pegboard corner to detect.

> **Theorem** HRW22, Theorem 27 (special class) Under the paper's stated hypotheses: M is compact, orientable, irreducible, has torus boundary and is a knot complement in an L-space. The following are equivalent: (i) M is a Heegaard Floer homology solid torus; (ii) every filling slope other than the rational longitude is an L-space; (iii) $\mathcal{C}(M)$ has trivial local systems and can be regularly homotoped into a small neighborhood of the rational longitude.

Example: an actual solid torus. $M = D^2 \times S^1$ has $\mathcal{C}(M)$ represented by one embedded curve of its disk-bounding meridian slope. Every filling except the slope producing $S^1 \times S^2$ gives a lens space (including S^3), hence an L-space. It has a completely different "all but one slope'' interval pattern from a non-solid-torus-like Floer-simple exterior. The boundary-incompressibility hypothesis of HRW23, Theorem 1.14 also fails for a genuine solid torus.

Diagnostic distinctions: A single L-space filling does not imply Floer simplicity. Being loop type does not imply Floer simplicity (HRW22, Section 1.3 gives examples with no L-space fillings). Two-valued torsion is a *consequence* of the graph theorem, not a general substitute for checking the graph theorem's assumptions. An arbitrary abstract curve object is not necessarily the invariant of a three-manifold; HRW23 explicitly distinguishes the extendable algebraic category from the manifold-derived subclass.

11. Reading assignment and proof checkpoints

Study allocation (4 hours 50 minutes):

Time	Close reading and active work	What must be reproduced
45 min	HRW23, Section 7.5, Theorem 7.17, with its proof	Definition of singular slope; both implications of the nonsingular filling criterion
50 min	HRW22, Section 5.3, Lemma 66 and proof	Why an interior L-space reference slope forces a graph
50 min	HRW22, Proposition 67 and its pegboard discussion	Both black-white inclusion and extremal-corner inclusion
40 min	HRW22, Lemma 68 and Theorem 69	Affine torsor quotient, full preimage, projectivization, interval selection
45 min	HRW23, Theorem 1.14 and Section 7.6	How L-space intervals combine under torus gluing; explicit proof-sketch limits

Time	Close reading and active work	What must be reproduced
60 min	Exercises 2, 4, 5, 7, and oral questions	Computations with slope determinants, graph derivatives, and two-arc coverage

Proof checkpoints

- ☐ Explain $\mathcal{L}(M)$ versus $\mathcal{L}^{\circ}(M)$ in the projective slope circle.
- ☐ State HRW23, Theorem 7.17, and say why a singular tangent is not just an extra intersection in an arbitrary drawing.
- ☐ Explain the role of the **interior** hypothesis in HRW22, Lemma 66.
- ☐ Derive black/white $\tau = 0/1$ from signed rays and the graph condition.
- ☐ State the order constraint in $X_{\mathfrak{s}}$ without reversing black and white.
- ☐ Prove the black-white direction gives a tangent using the two-intersection/Rolle argument.
- ☐ Explain the pegboard extremal-corner step in Proposition 67, explicitly marking it as diagram-dependent.
- ☐ Prove HRW22, Lemma 68 using the relative-to-absolute Spin^c quotient.
- ☐ State and assemble HRW22, Theorem 69 with **all** its hypotheses and the separate endpoint caveat.
- ☐ State HRW23, Theorem 1.14, and check its boundary-incompressibility hypothesis.

12. Eight qualifying-exam exercises

Exercise Exercise 1 - Conventions and complete hypotheses State definitions of $\mathrm{Sl}(\partial M)$, $\mathcal{L}(M)$, $\mathcal{L}^{\circ}(M)$ and Floer simplicity. State HRW22, Theorem 69 **without omitting** the condition on solid-torus-like manifolds or the known reference slope. Then write two sentences explaining why its conclusion is not a statement that every endpoint belongs to $\mathcal{L}(M)$.

Hint. Compare the exact published statement at the end of Section 5.3 with HRW23, Theorem 7.17.

Exercise Exercise 2 - Prove the graph lemma from Floer pairing Let $\alpha \in \mathcal{L}^{\circ}(M)$ and use coordinates (3.1). Explain how minimal intersection of the vertical test circle with every relevant Spin^c-refined component implies a single-valued graph. Identify the one place where $\alpha \in \mathcal{L}^{\circ}$, rather than just $\mathcal{L}$, is used. Then say why arbitrary nontrivial rank-r local systems cannot simply be ignored in the intersection count.

Hint. Use HRW22, Lemma 66, and the slope-singularity theorem.

Exercise Exercise 3 - A rigorous mean-value calculation For $f(x) = x/2 + \sin(2\pi x)/16$, determine the exact interval of slopes of its tangent lines. Find all $x \in [0, 1)$ where the tangent has slope $1/2$. Show that a line of slope 2 cannot be tangent. Explain why this *does not* compute $\mathcal{L}^{\circ}$ of a three-manifold.

Hint. Differentiate and use $-1 \le \cos\theta \le 1$.

Exercise Exercise 4 - Relative Spin^c preimages and projectivization For $j_* : \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}/3$ given by $(a, b) \mapsto (a, [b]_3)$, compute the full preimages of $(2, [1]_3)$ and $(1, [0]_3)$. Projectivize their nonzero vectors. Identify the kernel and explain why taking one arbitrary lift of a class in $H_1(M)$ gives an incomplete forbidden-slope set. Which of your vectors are nonprimitive?

Hint. Solve one integer equation and one congruence.

Exercise Exercise 5 - The lattice proof of Lemma 68 Starting from Spin$^c(M, \partial M)$ as a torsor for $H_1(M)$, prove that two relative Spin^c classes restrict to the same absolute class if and only if their difference lies in $\operatorname{im} j_*$. Translate each condition in the definition of $D^{\tau}(M)$ into a statement about two punctures in the same lifted plane, and deduce (6.3).

Hint. Use exactness of the relative/absolute grading action and Proposition 62. State clearly why one takes a union over $\mathfrak{s}$.

Exercise Exercise 6 - Determinants, Floer dimensions, and canceling pairs Calculate $i_{\min}((2,3),(1,4))$, $i_{\min}((3,7),(2,5))$, and $i_{\min}((1,0),(3,7))$. For straight embedded essential slope circles with trivial rank-one local systems, determine the Floer dimensions. Separately prove $\dim H(C) = \dim C - 2\operatorname{rank} d$ for a finite-dimensional complex over $\mathbb{F}_2$. Why can the raw number of intersections in a nonminimal picture exceed the Floer dimension?

Hint. Determinants for the minimal positions; rank-nullity for the complex.

Exercise Exercise 7 - A torus gluing test State HRW23, Theorem 1.14 precisely. Regard the slope circle abstractly as $\mathbb{R}/\mathbb{Z}$ and test whether the two collections of open arcs cover the circle: (i) $(0.1, 0.8)$ and $(0.6, 1.2)$ modulo 1; (ii) $(0.1, 0.6)$ and $(0.5, 0.9)$. Explain why *overlap* is not enough. Explain why you cannot apply the theorem unchanged when one piece is a genuine solid torus.

Hint. Draw intervals on a circle and inspect the gaps.

Exercise Exercise 8 - A source-dependency proof audit Build a dependency diagram for the following results: HRW23 Theorems 1.1, 1.2, 7.17, 1.14; HRW22 Proposition 62, Lemma 66, Proposition 67, Lemma 68, Theorem 69. Mark which implications depend on nontrivial local-system information, which depend on the Floer-simple hypothesis, and which are special to actual three-manifold invariants. Explain why the 2022 result does not apply to arbitrary extendable type D modules.

Hint. Distinguish algebraic classification from the restricted manifold-derived subclass and mark at least two places where the full 2023 paper is needed.

13. Solutions and detailed hints

Solution 1

The slope circle is $\mathbb{P}(H_1(\partial M;\mathbb{Q})) \cong \mathbb{Q}P^1$. The L-space slope set consists of primitive slopes giving rational-homology-sphere fillings with $\dim \widehat{HF} = |H_1|$. Floer simple means nonempty *interior* $\mathcal{L}^\circ$. Theorem 69 requires Floer simple, not solid-torus-like, and an identified α in the interior; it returns the maximal open projective interval through α disjoint from the projected preimage of D^τ. Boundary slopes are not determined merely by taking closure of the interior; tangencies can occur at the endpoints and the closed-manifold L-space condition must be checked there.

Solution 2

By HRW23's filling pairing, $\widehat{HF}(M(\alpha))$ is computed by pairing $\mathcal{C}(M)$ with a straight slope-α circle. The L-space rank condition implies minimal intersection in every relevant grading sector. With α vertical, a generic vertical line meets the lifted curve once, hence projection onto x is injective and surjective on generic fibers. The only possible failure of a graph consists of vertical segments through pegs. By HRW23, Theorem 7.17, α interior forbids essential tangent segments, so they cannot occur. A rank-r local system contributes r-dimensional fibers at intersections; the counting argument must use those multiplicities, rather than treating all intersections as one-dimensional.

Solution 3

Differentiate:

$$f'(x) = \frac{1}{2} + \frac{\pi}{8}\cos(2\pi x).$$

The image of f' is $[1/2-\pi/8,\ 1/2+\pi/8]$. The derivative equals $1/2$ exactly when $\cos(2\pi x) = 0$, i.e. $x = 1/4, 3/4$ modulo integers. Since $2 > 1/2 + \pi/8$, slope 2 cannot occur. The object is only an analytic toy: it has not been shown to be an HRW train track arising from a three-manifold, to obey lattice pegboard tightness, or to carry an L-space reference filling. Hence no theorem identifies these derivative bounds with an L-space interval.

Solution 4

Solve $a = 2$, $b \equiv 1 \pmod 3$ to get

$$j_*^{-1}(2,[1]) = \{(2, 1+3k) : k \in \mathbb{Z}\}.$$

Similarly

$$j_*^{-1}(1,[0]) = \{(1, 3k) : k \in \mathbb{Z}\}.$$

Projectivization gives $\{[2 : 1+3k]\}$ and $\{[1 : 3k]\}$ respectively. The kernel is $\{(0, 3k) : k \in \mathbb{Z}\}$. The vectors $(2, 1+3k)$ are primitive exactly when $1+3k$ is odd; $(1, 3k)$ are always primitive. Choosing one representative from either fiber would omit almost every slope direction. Torsion classes $[b]_3$ determine infinitely many integral lifts; the kernel cannot be dropped.

Solution 5

The relative grading torsor is an affine $H_1(M)$-space. Forgetting the boundary condition gives the absolute Spin^c torsor, and the quotient of its difference group is $H_1(M)/\operatorname{im} j_*$. Hence relative differences lying in $\operatorname{im} j_*$ are precisely those with equal absolute restrictions. For such a pair, choose punctures p_0, p_1 in the same grading lift. The intersection theorem (HRW22, Proposition 62) converts torsion 0 to black and torsion 1 to white. The longitude order inequality is exactly the condition imposed on their vector difference. All preimages are realized by deck-translated puncture pairs, so their directions form $X_{\mathfrak{s}}$. Conversely each black-white pair gives an admissible relative difference. Taking all $\mathfrak{s}$ yields Lemma 68.

Solution 6

The determinants are

$$|2 \cdot 4 - 3 \cdot 1| = 5, \qquad |3 \cdot 5 - 7 \cdot 2| = 1, \qquad |1 \cdot 7 - 0 \cdot 3| = 7.$$

For straight embedded, nonparallel, rank-one, trivial-local-system circles in minimal position, there are no bigons, so the corresponding Floer dimensions are $5, 1, 7$. Independently, $\dim \ker d = \dim C - \operatorname{rank} d$ and $\dim H = \dim \ker d - \dim \operatorname{im} d$, proving (8.2). In a nonminimal drawing a Reidemeister-II-type finger move may create a pair of intersections joined by a differential bigon, so the number of generators increases by two while homology remains unchanged.

Solution 7

For boundary-incompressible torus-boundary M_0, M_1, HRW23, Theorem 1.14 says the glued manifold is an L-space if and only if the union of the two *interior* L-space slope sets, after transporting one by the gluing map, covers all rational slopes. For (i), $(0.1, 0.8)$ covers the central arc and $(0.6, 1.2)$ covers its complement around $0 \sim 1$; together they cover the circle. For (ii), the interval from 0.9 back to 0.1 is uncovered. Nonempty overlap merely means some slope is good on both sides, not that every slope is good on at least one side. A solid torus is boundary-compressible, so Theorem 1.14's hypotheses fail; the relevant solid-torus filling pairing must be used instead.

Solution 8

HRW23 Theorem 1.1 supplies the manifold's invariant from bordered type D structures. Theorem 1.2 identifies torus gluing and Dehn filling with Lagrangian Floer pairings. Theorem 7.17 turns minimal rank into the complement of essential-tangent slopes; the proof requires the manifold curve constraints and admissible geometric pairing. HRW22 Proposition 62 then identifies its oriented grading-cover ray counts with Turaev torsion. **Floer simplicity** and an interior filling are used in Lemma 66 to obtain a graph. In Proposition 67 the graph is colored and its extreme slopes are read from black-white pegs; the non-solid-torus-like hypothesis controls genuine corners. Lemma 68 translates these pegs to the algebraic D^τ data. Theorem 69 assembles the conclusions. HRW23 Theorem 1.14 supplies an additional gluing application under boundary incompressibility. The first 2023 structure theorem classifies many abstract extendable type D modules; not every such object comes from a manifold, so the torsion/filling hypotheses cannot simply be imported into that larger category.

14. Oral exam questions and model answers

Question Q1. What makes the α-adapted lift a graph? **Model answer.** HRW23's pairing identifies $M(\alpha)$ with intersections against a straight slope circle. An L-space filling forces the minimal possible intersection rank. For generic vertical lines each Spin^c lift therefore has one intersection. A potential vertical segment would make α singular, which is ruled out by $\alpha \in \mathcal{L}^\circ(M)$. This is HRW22, Lemma 66.

Question Q2. How does torsion become a black-white coloring? **Model answer.** HRW22, Proposition 62 expresses torsion as algebraic intersection of a downward puncture ray with the oriented curve. The curve is a graph and divides the plane; a ray from the negative side has intersection zero, and a ray from the positive side has intersection one. Hence black pegs are torsion zero and white pegs torsion one.

Question Q3. Why is the black-white order condition essential? **Model answer.** The condition $l \cdot (p - q) \geq 0$, with p black and q white, ensures that the ray through the two pegs has the asymptotic behavior used to obtain *two* intersections with the graph. A first crossing alone proves no tangency. The second crossing allows the mean-value theorem to force a tangent of direction $[p - q]$.

Question Q4. What exactly is the difference between D^τ and the set of forbidden slopes? **Model answer.** D^τ lies in $H_1(M)$ and consists of ordered differences of relative Spin^c structures with torsion zero and one. A boundary slope is an element of $\mathbb{P}(H_1(\partial M))$. One must take **all** preimages of D^τ under the boundary inclusion map and then projectivize the nonzero vectors. Lemma 68 identifies these directions with black-white peg slopes.

Question Q5. State HRW22 Theorem 69 and its endpoint limitation. **Model answer.** If M is Floer simple, not solid-torus-like, and α is a known interior L-space slope, then $\mathcal{L}^\circ(M)$ is the largest projective slope interval containing α and avoiding the projectivized preimage of D^τ. This computes the **interior**, not the membership of every boundary slope of its closure.

Question Q6. How does HRW23 Theorem 7.17 differ from Theorem 69? **Model answer.** Theorem 7.17 is a geometric criterion for the interior of the L-space set in terms of essential tangent slopes,

and does not require first computing torsion. Theorem 69 is the Floer-simple torsion reconstruction of that tangent interval, with an extra non-solid-torus-like condition and a chosen interior reference slope.

Question Q7. What is the precise L-space gluing test? **Model answer.** Under the boundary-incompressibility hypotheses of HRW23, Theorem 1.14, $Y = M_0 \cup_h M_1$ is an L-space precisely when $\mathcal{L}^\circ(M_0) \cup h(\mathcal{L}^\circ(M_1))$ is **every** rational slope. Intersecting intervals are not enough: their union must cover the projective circle.

Question Q8. Why is the solid torus an exception? **Model answer.** Its immersed curve has no genuine corners and can be placed near the disk-bounding longitude/meridian according to the marking. Every filling other than the one producing $S^1 \times S^2$ is a lens space. The extremal-corner proof of Proposition 67 does not apply and the boundary-incompressibility hypothesis of the gluing theorem also fails.

15. Exact references and next lecture

[**HRW22**] Jonathan Hanselman, Jacob Rasmussen, Liam Watson, *Heegaard Floer homology for manifolds with torus boundary: properties and examples*, *Proceedings of the London Mathematical Society* **125** (2022), 879-967, DOI 10.1112/plms.12473. Exact reading: published **Section 5.3**, **Lemma 66** (graph condition), **Proposition 67** (black-white slopes), **Lemma 68** (torsion-difference identification), **Theorem 69** (interval). Supporting: **Section 5.1, Proposition 62** (torsion and signed ray intersections); **Section 1.5, Theorem 27** (Heegaard Floer homology solid tori). Published full text. Note: Theorem 69 = Theorem 68 in arXiv:1810.10355v1; we follow the published numbering.

[**HRW23**] Jonathan Hanselman, Jacob Rasmussen, Liam Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, arXiv:1604.03466v3 (2023). Exact reading: **Theorems 1.1, 1.2, and 1.14** in the introduction; **Section 7.5, Theorem 7.17** (L-space interior and essential tangencies); **Section 7.6** (L-space gluing), and **Section 6** for the grading-cover background. The 2023 source is the three-author arXiv paper; do not cite the distinct two-author 2023 *A calculus for bordered Floer homology* as this paper.

[**RR17, supporting**] Jacob Rasmussen and Sarah Dean Rasmussen, *Floer Simple Manifolds and L-Space Intervals*, *Advances in Mathematics* **322** (2017), 738-805, DOI 10.1016/j.aim.2017.10.014. HRW22, Theorem 69 cites its Theorem 1 as the algebraic torsion interval characterization.

[**LOT18, background only**] Robert Lipshitz, Peter Ozsv'ath, Dylan Thurston, *Bordered Heegaard Floer Homology*, *Memoirs of the AMS* **254** (2018), no. 1216, DOI 10.1090/memo/1216. Use only for the boxed tensor product and torus-boundary differential module formalism.

Day 13 preview. We move from filling intervals to the *two-piece gluing problem* in HRW23, Sections 7.5-7.7 and HRW22, Section 1.5. We will compare the exact L-space gluing criterion with minimal intersection inequalities, analyze why boundary incompressibility is needed, and compute simple toroidal gluing examples without substituting abstract interval diagrams for actual manifold calculations.