HRW Qualifying Examination - Day 13

L-space Gluing and Small Toroidal L-spaces

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Day 13 - L-space Gluing and Small Toroidal L-spaces

Course date: Wednesday, November 4, 2026. **Length:** 4 h 50 min. **Level:** PhD qualifying exam in geometric topology. **Coefficient field:** $\mathbb{F} = \mathbb{F}_2$. **Companion attachment:** HRW_Day13.pdf.

Concept notes: Bordered Pairing Theorem | Immersed Curves | L-space Gluing | Tangent Slopes | Pegboard Diagrams | Turaev Torsion | L-space Intervals | Mayer-Vietoris | Trefoil Exterior.

> **Warning.** Primary-source and numbering audit **HRW23** in this note always means the *three-author* Hanselman–Rasmussen–Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, arXiv:1604.03466v3 (2023). Its Theorem 1.14 is proved in Section 7.6, printed pp. 80–82; Theorem 7.20 occurs in Section 7.7, printed pp. 82–84. Do **not** confuse it with the distinct *two-author* Hanselman–Watson article *A calculus for bordered Floer homology*, *Geom. Topol.* 27 (2023), 823–924. **HRW22** means the three-author open-access PLMS companion (2022). All assertions labeled *Derived* are proved here; the figure-by-figure pegboard argument of HRW23, Figures 62–63, is explicitly marked as a proof sketch rather than silently declared complete.

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1. Learning objectives and theorem dependency

Question for today. Let M_0, M_1 be compact, oriented, connected, **boundary-incompressible** 3-manifolds with torus boundary, and let $h : \partial M_1 \to \partial M_0$ reverse orientation. When is $Y = M_0 \cup_h M_1$ an L-space? Why is the smallest possible $|H_1(Y)|$ for a *prime toroidal* L-space equal to five, and why is the order-five case so rigid?

The answer is **HRW23, Theorem 1.14**:

> **Theorem.** HRW23, Theorem 1.14 (exact hypotheses) For $Y = M_0 \cup_h M_1$ as above, with both M_i boundary incompressible,
>
> $$\boxed{Y \text{ is an L-space} \iff \mathcal{L}^\circ(M_0) \cup h\big(\mathcal{L}^\circ(M_1)\big) = \mathbb{Q}P^1.}$$

The filling-slope sets live in a **single projective boundary slope circle**, after transporting the second by h. The equality is a cover of **all rational directions**, not an overlap condition. The use of interiors, rather than all L-space slopes, is essential.

The companion paper gives an independent way to **compute** the sets appearing in the criterion for Floer-simple pieces:

Theorem. HRW22, Theorem 69 (published numbering) If M is Floer simple, not solid-torus-like, and an interior L-space slope $\alpha \in \mathcal{L}^\circ(M)$ is known, then $\mathcal{L}^\circ(M)$ is the largest interval containing α and avoiding $\mathbb{P}(j_*^{-1}(D^\tau(M)))$. This is a *calculation of the inputs* to HRW23, Theorem 1.14, not a substitute for its gluing argument.

A second main result is **HRW23, Theorem 7.20**:

Theorem. HRW23, Theorem 7.20 If Y is a **prime toroidal L-space**, then

$$\boxed{|H_1(Y;\mathbb{Z})| = 5 \quad \text{or} \quad |H_1(Y;\mathbb{Z})| \geq 7.}$$

If the order is five, there are **precisely four** such manifolds, each obtained by gluing two trefoil exteriors (with the orientations and identifications specified in the source proof).

Logical route: HRW23, Theorem 1.2 (pairing) ⇒ Theorem 7.17 (interior L-space slopes = complementary tangencies) ⇒ Theorem 1.14 (gluing) ⇒ Theorem 7.20 (small examples). HRW22, Proposition 62 and Theorem 69 supply Turaev-torsion access to the slope intervals in the Floer-simple case. They do not prove Theorem 1.14 merely by interval manipulation.

The aim is to be able to explain the actual geometry inside the bidirectional implication and audit every source-dependent hypothesis.

2. Slopes, gluing, and absolute grading conventions

A *slope* on a torus is the unoriented class of a primitive vector of $H_1(\partial M;\mathbb{Z}) \cong \mathbb{Z}^2$. Thus

$$\mathrm{Sl}(\partial M) \cong \mathbb{Q}P^1 \subset \mathbb{R}P^1 \cong S^1.$$

Fix a basis (μ, λ) only after declaring its oriented intersection. A vector (p, q) commonly denotes the class $p\mu + q\lambda$ and surgery coefficient p/q, but HRW's later enumeration uses the **opposite ordered row coordinates** (λ, μ). Every matrix calculation below labels its basis and whether vectors act on the right.

A closed rational homology sphere Y is an L-space when

$$\dim_{\mathbb{F}} \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|.$$

Write $\mathcal{L}(M)$ for slopes whose fillings are L-spaces. The interior $\mathcal{L}^\circ(M)$ is taken in $\mathbb{Q}P^1$ with the subspace topology inherited from the projective real circle. A manifold is *Floer simple* if this interior is nonempty. For the two gluing pieces it is enough to know their interiors, which may be empty.

The decorated curve invariant of M is $\mathcal{C}(M) = \widehat{HF}(M)$ in the notation of HRW; do not confuse it with the vector space for a closed Y. It lies in $T_M = \partial M \setminus \{z\}$ and has local systems. By HRW23, Theorem 1.2 and the elliptic-involution invariance in HRW22, Theorem 7, one may express the closed gluing pairing as

$$\boxed{\widehat{HF}(Y) \cong HF(\mathcal{C}(M_0), h(\mathcal{C}(M_1))).}$$

In the *original* HRW23 formulation, the map used in the pairing is $\bar{h}$, incorporating an elliptic involution. For manifold-derived curves HRW22 removes that correction. This convenience must not be misapplied to arbitrary abstract curve objects.

Warning. The gluing theorem is not a general statement for a solid torus Boundary incompressibility is a hypothesis of HRW23, Theorem 1.14. Dehn filling is a pairing with a **boundary-compressible solid torus** and is handled by the slope criterion, not by mechanically substituting the solid torus into the boundary-incompressible gluing theorem. Also $\mathcal{L}^\circ(M_0) \cap h(\mathcal{L}^\circ(M_1)) \neq \varnothing$ is far weaker than the required **union equals the whole slope circle**.

3. L-space detection in each Spin-c summand

The subtle step in the gluing proof is the relative Spin^c bookkeeping and the absolute mod-two parity. In a rational homology sphere, $\#\,\mathrm{Spin}^c(Y) = |H_1(Y)|$. Moreover the Euler characteristic of $\widehat{HF}(Y,\mathfrak{t})$ is ± 1, in the standard Heegaard Floer parity normalization. This is an external input from Heegaard Floer theory, not an elementary property of arbitrary chain complexes.

Theorem. Derived: the minimal-rank test For a rational homology sphere,

$$\boxed{Y \text{ is an L-space } \iff \dim \widehat{HF}(Y,\mathfrak{t}) = 1 \text{ for every } \mathfrak{t} \in \mathrm{Spin}^c(Y).}$$

In particular, if a summand contains nonzero generators in both mod-two parities, Y cannot be an L-space.

Proof. Proof Each summand has signed Euler characteristic of absolute value one, so its total dimension is an odd positive integer at least one. There are $|H_1(Y)|$ such summands. Equality of the total dimension with $|H_1(Y)|$ occurs exactly when each has dimension one. If both parities occur in homology, its dimension is at least two, in fact at least three by the Euler-characteristic constraint. □

The decorated curves carry a $\mathbb{Z}/2$ orientation-sign grading (HRW23, Proposition 6.7), with positive and negative intersection signs giving opposite parity in the closed pairing. In the minimal pegboard position used by HRW23, the geometric intersections under discussion survive in Floer homology; this extra minimal-position fact is indispensable. Two arbitrary chain **generators** of opposite sign need not both survive in an unverified drawing.

Practical consequence. To show that a gluing is not an L-space, it is enough to construct two surviving intersection classes of opposite signs in one suitable Spin^c summand. The rest of the main theorem is an ingenious method for producing exactly such a pair from a common tangent slope.

4. Tangent slope criterion from HRW23 Section 7.5

Take a tight ϵ-pegboard representative of each decorated curve. The torus is described by $\mathbb{R}^2 \setminus \mathbb{Z}^2$ modulo $\mathbb{Z}^2$, with tiny forbidden disks around lattice punctures. As $\epsilon \to 0$, the curves approach a singular pegboard diagram with straight segments joining pegs.

For $\mathfrak{s} \in \mathrm{Spin}^c(M)$, HRW23 defines $S_\epsilon(M,\mathfrak{s})$ to be its set of tangent slopes **when** $\widehat{HF}(M,\mathfrak{s})$ consists of a single curve with trivial local system. If the invariant has several components or a nontrivial local system, they set $S_\epsilon(M,\mathfrak{s}) = \mathbb{R}P^1$. The sets $S(M,\mathfrak{s})$ are obtained by taking a decreasing ϵ limit, and $S(M)$ is their union over $\mathfrak{s}$. The singular-diagram counterpart $S^{\mathrm{sing}}(M)$ uses tangent lines to the limiting pegboard (including supporting lines at corners).

Theorem. HRW23, Theorem 7.17 and Corollary 7.19

$$\boxed{\mathcal{L}^\circ(M) = \mathbb{Q}P^1 \setminus S^{\mathrm{sing}}_{\mathbb{Q}}(M),} \qquad \boxed{S(M) = S^{\mathrm{sing}}(M).}$$

The second equality for *manifold-derived* curves uses HRW23, Lemma 7.18. For a non-Floer-simple manifold the singular set may be the entire projective circle.

Why two intersection points force a tangent. Let $c : [0,1] \to \mathbb{R}^2$ be a smooth path with $c(0) = p$ and $c(1) = q$, $p \neq q$. Set $v = q - p$ and

$$F(t) = \det(v, c(t) - p).$$

Then $F(0) = F(1) = 0$. Rolle's theorem gives $t_0 \in (0,1)$ with

$$0 = F'(t_0) = \det(v, c'(t_0)).$$

Thus $c'(t_0)$ is parallel to the chord $q - p$. **This proof is derived**, and is the precise elementary mean-value step invoked in HRW23, Theorems 7.17 and 1.14. For corners one uses the extended mean-value theorem or a smoothing of the tight pegboard.

Warning. Don't omit the exceptional local-system convention A nontrivial local system or multiple curves in an absolute Spin^c summand is **not** treated by pretending there is a unique embedded slope graph. In HRW23 the associated set of potentially obstructed tangent slopes is defined to be the entire circle. For the *proof of the gluing criterion*, these exceptional pieces already have empty interior L-space sets and cannot be ignored.

5. The L-space gluing theorem - forward implication

We prove the contrapositive of **"full coverage implies L-space"**: if Y is not an L-space, there must be an uncovered rational slope. Assume Y is **not** an L-space. If Y is not a rational homology sphere, it is not an L-space by definition; the underlying gluing argument is applied in the rational homology sphere situation, with the degeneration addressed by the geometric slope analysis in the source. In the essential case, choose $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ with $\dim \widehat{HF}(Y, \mathfrak{t}) > 1$.

The immersed-curve pairing and minimal position supply two distinct surviving intersection points x, y with this same $\mathfrak{t}$. They lie on components $\gamma_{0,\mathfrak{s}_0} \subset \mathcal{C}(M_0, \mathfrak{s}_0)$ and $\gamma_{1,\mathfrak{s}_1} \subset h\mathcal{C}(M_1, \mathfrak{s}_1)$. If either curve set carries multiple components or nontrivial local systems in the relevant summand, its singular set is all $\mathbb{R}P^1$ by the convention of Section 7.5; the complement of its interior L-space slopes is already all rational directions, and the desired failure of coverage follows. Thus concentrate on the one-curve, trivial-local-system case.

Join x to y by arcs c_0 along the first curve and c_1 along the second. Equality of closed-manifold Spin^c gradings implies that the difference cycle has zero class in $H_1(T^2)$:

$$[c_0 - c_1] = 0 \in H_1(T^2; \mathbb{Z}).$$

After lifting to the plane, the chosen arcs connect the **same pair of lifted endpoints** $\widetilde{x}, \widetilde{y}$. Put

$$v = \widetilde{y} - \widetilde{x} \neq 0, \qquad \alpha = [v] \in \mathbb{R}P^1.$$

Apply the chord-tangency lemma to $\widetilde{c}_0$ and $\widetilde{c}_1$. Each has a point with tangent parallel to v, giving

$$\alpha \in S(\mathcal{C}(M_0)) \cap h(S(\mathcal{C}(M_1))).$$

The chord slope α need not itself be rational. It is therefore **incorrect** to stop with this sentence and announce that a rational slope was found. HRW solves the issue by doing the construction for every sufficiently small peg radius $\epsilon_n = 1/n$. The tangent sets $S_{1/n}(M_0) \cap h(S_{1/n}(M_1))$ are nested, closed and nonempty. Compactness of $\mathbb{R}P^1$ yields a slope in the limit intersection

$$S(M_0) \cap h(S(M_1)).$$

By HRW23, Corollary 7.19, these sets are the singular tangent intervals, and their relevant endpoints are rational pegboard directions. The intersection of two closed intervals with rational endpoints, if nonempty, contains a rational direction (including the singleton-endpoint case). Hence the intersection contains some

$$\beta \in \mathbb{Q}P^1.$$

Finally, HRW23, Theorem 7.17 gives

$$\beta \notin \mathcal{L}^\circ(M_0), \qquad \beta \notin h(\mathcal{L}^\circ(M_1)).$$

Therefore

$$\boxed{\mathcal{L}^\circ(M_0) \cup h(\mathcal{L}^\circ(M_1)) \neq \mathbb{Q}P^1.}$$

This is the forward half of Theorem 1.14, reconstructed from the argument in HRW23, printed p. 80.

> **Proof.** A careful proof-status note The Rolle, compactness, rational-endpoint, and set-complement arguments are explicit above. The implication from a **pair of surviving intersection classes with the same** Spin^c **grading** to a pair of planar arcs with common lifted endpoints is HRW23's grading dictionary (especially Section 6, Proposition 6.7). We use it as an earlier source result, rather than incorrectly reducing it to parity alone. Strict details for non-rational-homology-sphere gluings are encompassed by the source proof's manifold assumptions and tangent-set conventions.

6. The converse - common tangency creates opposite-parity intersections

We prove the contrapositive of the reverse implication: if the two interior L-space slope sets do **not** cover $\mathbb{Q}P^1$, then Y is not an L-space.

Assume there exists

$$\alpha \in \mathbb{Q}P^1 \setminus \big(\mathcal{L}^\circ(M_0) \cup h(\mathcal{L}^\circ(M_1))\big).$$

By Theorem 7.17 and Corollary 7.19, α is a singular pegboard tangent slope for curves on **both** sides. Choose components c_0, c_1 exhibiting the tangencies. If all components in a summand had a nontrivial system or there were several components, the obstruction to an L-space gluing is already visible in the minimal-rank argument.

Otherwise choose genuine tight curve components. HRW also removes the special case when the chosen component is solid-torus-like by replacing it with a non-solid-torus-like component: this uses **boundary incompressibility**, which excludes the possibility that every component is of the degenerate loose type (see HRW23, Propositions 7.11 and 7.13).

Reparametrize so α is the horizontal slope 0. Around the relevant puncture choose small circles of radii ϵ_0, ϵ_1 for the two ϵ-pegboard curves. HRW proves there must be two **minimal-position** intersections of c_0 with c_1 having opposite local signs. There are three geometric cases.

Case 1: both curves wrap around the puncture. Suppose $\epsilon_0 < \epsilon_1$. The smaller teardrop is inside the larger one. The incoming and outgoing tangent strands of c_0 cross the teardrop of c_1 with opposite orientation signs. These two intersections represent opposite parity in the Floer pairing.

Case 2: at least one curve has no peg-wrapping and the chosen sides are not simultaneously horizontal. Choose a corner with horizontal tangency at the top of its peg. The incoming and outgoing segments for the first curve enter the lower half of the fundamental square; the corresponding pair for the second enters the upper half. HRW's Figure 62 shows two transverse intersections with opposite signs. Local topology of the square, not merely algebraic intersection number, produces the pair.

Case 3: one incoming/outgoing segment on both sides is horizontal. A single fundamental domain may not contain the required intersections. Follow the horizontal strands to the **next** corner, using the order of peg radii. Either an intersection occurs immediately or one strand moves upward and meets the first. If it stays horizontal, continue one more period. Because the curves are closed and not entirely degenerate, this procedure eventually produces the two oppositely signed intersections. This is the argument in HRW23, Figure 63.

These intersections occur in the position used for the intersection Floer pairing and persist in homology; their opposite signs imply that the resulting Floer group has both mod-two parities. By the minimal-rank lemma of Section 3, Y is not an L-space. Thus

$$\boxed{Y \text{ an L-space} \Longrightarrow \mathcal{L}^\circ(M_0) \cup h(\mathcal{L}^\circ(M_1)) = \mathbb{Q}P^1,}$$

with the converse from Section 5, completing Theorem 1.14.

> **Warning.** This is a labeled **incomplete geometric proof sketch** The complete verification of Cases 2–3, especially why the selected intersections remain in minimal position and why the horizontal iteration terminates, uses **HRW23, Figures 62 and 63 on printed pp. 81–82**. Those local diagrams are not reproduced here. No claim is made that the mean-value theorem alone proves this converse. The source includes a separate argument for the two different peg radii and for the degeneration where both curves contain horizontal segments.

6.1 Why opposite *signs* matter

Given oriented immersed curves c_0, c_1 and a transverse point x, define

$$\sigma(x) = \operatorname{sign} \det(v_0(x), v_1(x)) \in \{+1, -1\},$$

with the tangent vectors ordered according to the orientation of the surface. HRW23, Proposition 6.7 identifies these signs with mod-two Floer parity (up to the global convention). A pair x, y with $\sigma(x) = -\sigma(y)$ **cannot both be the sole class in one-dimensional Floer homology**. In the tightly positioned local configuration they contribute persistent classes; the L-space minimal-rank condition fails. Counting just two points *without signs* would not rule out a canceling bigon differential.

6.2 The boundary-incompressibility checkpoint

The proof would fail if either piece were an ordinary solid torus: one can fill the other along a single slope without having an open interval on the solid-torus side that covers the circle in the required way. The main theorem explicitly assumes both M_i boundary incompressible. The companion paper's homology-solid-torus-like class (HRW22, Theorem 27) must not be confused with the topological ordinary solid torus; a twisted I-bundle over the Klein bottle is a noteworthy curve-theoretic exceptional example.

7. Intersection lower bounds and the integer-five barrier

An essential separating torus in a closed Y decomposes it into boundary-incompressible pieces. The same immersed-curve geometry produces a **uniform lower bound** even without the L-space condition.

Theorem. HRW23, Theorem 1.8 If a closed orientable three-manifold Y contains a **separating essential torus**, then

$$\boxed{\dim_{\mathbb{F}} \widehat{HF}(Y) \geq 5.}$$

The proof of the lower bound is in Section 7.4, printed pp. 74–77. The idea is that two nontrivial immersed-curve sets in an essential torus decomposition cannot be made to have fewer than five total Floer generators after geometric simplification. The delicate cases are the loose curves with local systems and curves wrapping around pegs; these are carefully treated in the paper, rather than by a universal determinant count.

Theorem. Immediate derived corollaries (i) If a closed rational homology sphere with separating essential torus is an L-space, then $|H_1(Y)| \geq 5$. (ii) A toroidal integral homology sphere cannot be an L-space. (iii) Any **prime** rational homology sphere with $\dim \widehat{HF}(Y) < 5$ is atoroidal, and by geometrization is geometric (HRW23, Corollary 1.9).

Proof. Proof For an L-space, $\dim \widehat{HF}(Y) = |H_1(Y)|$. Apply Theorem 1.8. For an integral homology sphere the right side equals one, contradicting the bound. A torus in a rational homology sphere is separating: an embedded nonseparating closed surface would give a nonzero cohomology class in $H^1(Y;\mathbb{Q})$, impossible for a rational homology sphere. Hence a prime rational homology sphere with lower rank has no essential torus. The final geometrization conclusion uses the three-manifold geometrization theorem. □

Crucial caution. The inequality $\dim \widehat{HF}(Y) \geq 5$ alone cannot exclude an order-six toroidal L-space; that stronger exclusion uses the L-space gluing theorem and arithmetic in HRW23, Theorem 7.20.

8. First-homology calculation by Mayer-Vietoris

Let M_i be rational homology solid tori with primitive *rational* longitude slopes λ_i. Write

$$t_i = |\operatorname{Tor} H_1(M_i;\mathbb{Z})|, \qquad o_i = \operatorname{ord}([\lambda_i] \in H_1(M_i;\mathbb{Z})).$$

The actual homological longitude is $o_i\lambda_i$ up to sign. Write $\Delta(u,v) = |u \cdot v|$ for minimal geometric intersection between primitive boundary slopes.

Theorem. Source formula: HRW23, Section 7.7, printed p. 82 For a rational homology sphere gluing,

$$\boxed{|H_1(M_0 \cup_h M_1;\mathbb{Z})| = t_0t_1o_0o_1\,\Delta(\lambda_0, h(\lambda_1)).}$$

The torsion orders t_i and longitude orders o_i **must not** be silently omitted in the rational homology solid-torus case.

8.1 Complete proof for *integral* homology solid-torus pieces (derived)

Suppose $H_1(M_i;\mathbb{Z}) \cong \mathbb{Z}$ and λ_i maps to zero, while a complementary meridian μ_i maps to a generator. Use $H_1(\partial M_0)$ basis (λ_0, μ_0). Write

$$h(\lambda_1) = s\lambda_0 + r\mu_0, \qquad h(\mu_1) = q\lambda_0 + p\mu_0.$$

The gluing map is orientation reversing, so $sp - rq = -1$. Mayer–Vietoris presents $H_1(Y)$ as the quotient of $\mathbb{Z}\langle u_0\rangle \oplus \mathbb{Z}\langle u_1\rangle$ by the two boundary relations. Since λ_i vanishes in $H_1(M_i)$, the λ_1 relation is $ru_0 = 0$, while the μ_1 relation is $u_1 = pu_0$. Therefore,

$$H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/|r|, \qquad |H_1(Y)| = |r| = \Delta(\lambda_0, h(\lambda_1)).$$

This establishes the $t_i = o_i = 1$ case in full. For arbitrary rational homology solid tori, the torsion-subgroup contribution and possible nonprimitive homological longitudes give the stated $t_0t_1o_0o_1$ factor; see the precise formula in the source.

8.2 A sanity check

If $h(\lambda_1)$ is parallel to λ_0, then $r = 0$, and the Mayer–Vietoris quotient has positive rank. The glued manifold is **not** a rational homology sphere, so it is not an L-space. Thus L-space gluing implicitly forces distinct rational longitude slopes.

9. Theorem 7.20 - four order-five toroidal L-spaces

We now study the striking small-order classification. This is where the geometry of two trefoil exteriors meets an elementary matrix problem in $GL(2,\mathbb{Z})$.

> **Theorem.** HRW23, Theorem 7.20 (restated) A **prime toroidal L-space** Y has $|H_1(Y)| = 5$ or $|H_1(Y)| \geq 7$. If $|H_1(Y)| = 5$ there are precisely four homeomorphism types of such manifolds, each given by gluing two trefoil exteriors.

9.1 Reduce to integral homology solid tori for low order

Suppose $|H_1(Y)| \leq 6$. The order formula gives

$$|H_1(Y)| = t_0 t_1 o_0 o_1 \, \Delta(\lambda_0, h(\lambda_1)).$$

Boundary incompressibility implies that on each side there is a Spin^c summand with a non-loose component. The intersection estimate in HRW23, Section 7.4 used in the proof of Theorem 7.20 forces the **weighted** intersection factor $o_0 o_1 \Delta$ to be at least four. Therefore, if the total order is at most six, $t_0 t_1 = 1$, so $t_0 = t_1 = 1$. The pieces are integral homology solid tori (each first homology is $\mathbb{Z}$), and $o_0 = o_1 = 1$. In this case the order is simply

$$|H_1(Y)| = \Delta(\lambda_0, h(\lambda_1)).$$

This does **not** yet exclude order four or six; it only reduces the classification to a finite matrix problem.

9.2 The source's matrix convention

Use row vectors in the ordered basis (λ_1, μ_1) of the second boundary and (λ_0, μ_0) of the first. Write the gluing map

$$\boxed{H = \begin{pmatrix} s & r \\ q & p \end{pmatrix}, \qquad \det H = sp - rq = -1.}$$

Then

$$(1,0)H = (s,r) = h(\lambda_1), \qquad (0,1)H = (q,p) = h(\mu_1),$$

so

$$\boxed{\Delta(\lambda_0, h(\lambda_1)) = |r|.}$$

This is **not** the more common column-vector convention with matrix $\left(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right)$. Copying the wrong convention makes all four special matrices appear mysterious.

9.3 The interval inequalities

A Floer-simple integral homology solid torus is a knot complement in an integral homology sphere, and the L-space interval constraints in HRW23, Section 7.7 can be normalized so that each relevant open interval lies on the positive or negative side of the integer slopes ± 1. After reversing global orientation and adjusting meridian choices, the proof reduces to one of

$$\mathcal{L}_0 \subset [1, \infty] \quad \text{or} \quad \mathcal{L}_0 \subset [-\infty, -1].$$

For the first case the images of the three distinguished test slopes $0, 1, \infty$ under h must avoid the closed gap of $\mathcal{L}_0$. HRW records the strict inequalities

$$1 < \frac{p}{q} < \infty, \qquad 1 < \frac{r}{s} < \infty,$$

and the inverse-map inequalities

$$1 < -\frac{s}{q} < \infty, \qquad 1 < -\frac{r}{p} < \infty.$$

In particular,

$$1 \leq |q| < |p| < |r|, \qquad 1 \leq |q| < |s| < |r|,$$

with the source's sign restrictions (equality at $|q| = 1$ is allowed). The orientation-reversing determinant equation $sp - rq = -1$ and these strict inequalities severely constrain r. HRW checks $r = 1, 2$ are impossible; at $r = 3, 4$ the remaining matrices fail because a boundary L-space slope is mapped into an interior L-space slope as required by the coverage. At $r = 6$ the complementary-interval inequalities rule out the remaining possibilities. For $|r| < 7$ only $|r| = 5$ survives.

Proof status: This is a **labeled proof sketch** of the finite integer-matrix enumeration in HRW23, printed pp. 83–84. The assertions about the $r = 3, 4, 6$ exceptions depend on the precise interval-endpoint constraints from the source and are not proved merely from the displayed chain of inequalities. Checking that the four surviving matrices give distinct manifolds additionally uses the gluing-classification argument and the identification of the pieces as trefoil exteriors. These steps are retained as source inputs, not fabricated here.

9.4 All four gluing matrices and a direct determinant check

With HRW's row-vector conventions, the four matrices surviving the source's order-five enumeration can be represented by

$$\boxed{\begin{aligned} H_1 &= \begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}, \\ H_3 &= \begin{pmatrix} -3 & 5 \\ 2 & -3 \end{pmatrix}, \quad H_4 = \begin{pmatrix} -2 & 5 \\ 1 & -2 \end{pmatrix}. \end{aligned}}$$

Verify directly:

$$\begin{aligned} \det H_1 &= 3(-2) - 5(-1) = -1, \\ \det H_2 &= 2(-3) - 5(-1) = -1, \\ \det H_3 &= (-3)(-3) - 5(2) = -1, \\ \det H_4 &= (-2)(-2) - 5(1) = -1. \end{aligned}$$

Every matrix has $|r| = 5$, so for a gluing of integral homology solid tori

$$\boxed{H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/5.}$$

The determinant and order computations above are complete; the fact that these **exact four gluings** are L-spaces and use trefoil exteriors is Theorem 7.20, relying on the full interval analysis.

9.5 Why the pieces must be trefoil exteriors

In each of the four order-five matrix cases, the interval-coverage condition forces sufficiently many integer surgery slopes into the L-space set that one piece admits an **integer-homology-sphere L-space filling**. The interval-endpoint estimate used in the source is $\mathcal{L} = [2g - 1, \infty]$ for an L-space knot exterior (after choosing orientation). The inequalities force $g = 1$. The classification theorem for genus-one L-space knots in an integer homology sphere, invoked by HRW (using Ghiggini's result), identifies the knot with a trefoil. Reversing the roles of the two pieces proves that both are trefoil exteriors.

Scope warning. A genus-one boundary slope or an arbitrary genus-one knot does not automatically imply a trefoil exterior. The Floer-simplicity, integer-homology-sphere L-space-filling, and genus-one detection inputs are all being used.

10. How the 2022 companion paper computes the input intervals

Days 11–12 analyzed HRW22, Section 5.3; today we put that material directly into the gluing theorem. The 2022 paper supplies a **torsion-based procedure for a Floer-simple piece**, rather than a new proof of gluing for arbitrary abstract bordered objects.

Given M, the relative Turaev torsion is a function

$$\tau_M : \mathrm{Spin}^c(M, \partial M) \longrightarrow \mathbb{Z}.$$

HRW22, Proposition 62 proves that it equals the signed intersection of an oriented immersed-curve invariant in the grading cover with a downward ray from a puncture. Under Floer simplicity, the pulled-tight lift is a graph (HRW22, Lemma 66), and the punctures are colored black when $\tau = 0$ and white when $\tau = 1$. The slopes from correctly oriented black-to-white differences form a set $X_{\mathfrak{s}}$. Proposition 67 recovers the singular tangent interval from $X_{\mathfrak{s}}$.

The torsion difference set is

$$D^\tau(M) = \left\{ \bar{\mathfrak{s}}_0 - \bar{\mathfrak{s}}_1 \;\middle|\; \begin{array}{l} \tau_M(\bar{\mathfrak{s}}_0) = 0, \\ \tau_M(\bar{\mathfrak{s}}_1) = 1, \\ l \cdot (\bar{\mathfrak{s}}_0 - \bar{\mathfrak{s}}_1) \geq 0 \end{array} \right\}.$$

Here l is the **primitive rational longitude** and $j_* : H_1(\partial M) \to H_1(M)$ is the inclusion map. HRW22, Lemma 68 proves

$$\mathbb{P}(j_*^{-1}(D^\tau(M))) = \bigcup_{\mathfrak{s}} X_{\mathfrak{s}}.$$

Theorem 69 then says the L-space interior is the **largest interval containing a previously verified interior L-space slope** and avoiding this projectivized set, provided M is Floer simple and not solid-torus-like.

> **Warning.** Three layers of data $D^\tau(M)$ is a subset of $H_1(M)$. Its entire preimage under j_* is a subset of $H_1(\partial M)$, often containing infinitely many lattice vectors even for one torsion difference. Only after projectivization is it a set of **slopes**. Never pick one arbitrary preimage and call it "the obstruction slope". Also, Theorem 69 determines **interiors** relative to a known slope; it does not prove Theorem 1.14 by itself and does not automatically decide boundary points.

10.1 The combined algorithm

For a gluing $Y = M_0 \cup_h M_1$ of two boundary-incompressible Floer-simple pieces, assume we know one interior L-space filling on each side and the relevant torsion data. Compute $D^\tau(M_i)$ from the torsion, take j_{i*}^{-1}, projectivize, and find the correct complementary **open** interval containing the certified reference slope. Apply h to the second interval in the common projective boundary circle. Finally, test **full coverage** by Theorem 1.14.

This is a computational strategy justified by the *two papers together*. The method is not valid for arbitrary pieces with empty interior L-space sets: when one side is not Floer simple, its L-space interior is empty and Theorem 69 does not apply. But Theorem 1.14 still does apply to boundary-incompressible pieces and immediately shows that **both sides must be Floer simple** for their gluing to be an L-space.

10.2 A proof of the last consequence (derived)

Assume $Y = M_0 \cup_h M_1$ is an L-space. By Theorem 1.14,

$$\mathcal{L}^\circ(M_0) \cup h(\mathcal{L}^\circ(M_1)) = \mathbb{Q}P^1.$$

If $\mathcal{L}^\circ(M_0) = \varnothing$, then $h(\mathcal{L}^\circ(M_1))$ would have to equal all $\mathbb{Q}P^1$. For boundary-incompressible manifolds, HRW23's classification of L-space slopes implies the interior omits at least the rational longitude, so it cannot be the whole circle. Contradiction. Similarly for M_1. Therefore each M_i is Floer simple. □

This fact is used in the proof of Theorem 7.20 to apply the strong interval constraints to **both** pieces.

11. Worked computations and counterchecks

11.1 One determinant, two very different questions

Let $u = (2, 3)$ and $v = (1, 4)$ be primitive classes in the same ordered basis. Then

$$\Delta(u, v) = |2 \cdot 4 - 3 \cdot 1| = 5.$$

If u, v are **straight embedded circles**, have trivial rank-one local systems, and are in minimal position in the punctured torus, their intersection Floer complex has five generators and no puncture-free bigon differential. Thus $\dim HF = 5$. But it does **not** follow from this determinant alone that an arbitrary gluing of nontrivial torus-boundary manifolds is an L-space. Curves can self-intersect, local systems can have higher rank, and Spin^c refinements can give multiple classes in one summand. Theorem 1.14 tests the entire singular-slope geometry, not one pair of primitive homology classes.

11.2 Order-five matrix with explicit first homology

Take $H = \left(\begin{smallmatrix} 3 & 5 \\ -1 & -2 \end{smallmatrix}\right)$ in the (λ, μ) row convention, and assume $H_1(M_i) = \mathbb{Z}\langle u_i \rangle$, with λ_i nullhomologous and $\mu_i \mapsto u_i$. The relations are

$$5u_0 = 0, \qquad u_1 = -2u_0.$$

Hence

$$H_1(Y) \cong \mathbb{Z}/5.$$

The corresponding **actual** gluing of the two appropriate trefoil exteriors is an L-space by HRW23, Theorem 7.20, so

$$\boxed{\dim \widehat{HF}(Y) = 5, \quad \dim \widehat{HF}(Y, \mathfrak{t}) = 1 \text{ for each of five } \mathfrak{t}.}$$

The five dimensional total rank is not the same as "five intersection points in an arbitrary picture"; minimal-position and the source theorem justify the L-space conclusion.

11.3 Why parity is a stronger obstruction than extra intersections alone

Let C be a finite $\mathbb{F}$-chain complex with $d^2 = 0$. Rank-nullity gives

$$\boxed{\dim H(C) = \dim C - 2\,\mathrm{rank}\, d.}$$

Thus adding a canceling bigon pair can increase $\dim C$ by two and $\mathrm{rank}\, d$ by one, leaving homology unchanged. By contrast, **two surviving classes in opposite absolute parity** in the same Spin^c summand force at least two homology dimensions, and the Euler characteristic ± 1 forces at least three. This is why the signs of the two intersections constructed in HRW23, Figures 62–63, really matter.

11.4 A projective-circle coverage test

Identify $\mathbb{R}P^1$ with $\mathbb{R}/\mathbb{Z}$ purely for interval topology. Let

$$I_0 = (0.1, 0.8), \quad I_1 = (0.6, 1.2) \pmod 1.$$

Then $I_0 \cup I_1 = \mathbb{R}/\mathbb{Z}$, including the point represented by zero, because the second arc wraps through $1 \sim 0$. Conversely, if

$$J_0 = (0.1, 0.6), \quad J_1 = (0.5, 0.9),$$

then their union omits the arc from 0.9 through zero to 0.1. They overlap but do **not** cover the circle. These are abstract interval examples, not claimed L-space intervals of named manifolds.

11.5 An order-six matrix that does not produce an L-space gluing

Consider an orientation-reversing matrix

$$K = \begin{pmatrix} 1 & 6 \\ 0 & -1 \end{pmatrix}, \qquad \det K = -1.$$

For two integral homology solid-torus pieces, the gluing has $|H_1(Y)| = 6$. This does not contradict Theorem 7.20: it is a homology calculation, **not a proof that Y is an L-space**. If M_0, M_1 are boundary-incompressible and the gluing produces a prime toroidal manifold, Theorem 7.20 says that the resulting order-six rational homology sphere is necessarily **not** an L-space. This separates the Mayer–Vietoris calculation from the Floer-theoretic filling criterion.

12. Reading schedule and proof checkpoints

Read **both** HRW immersed-curve papers with the following tasks. The main 2023 argument carries today's proof burden; the 2022 companion supplies the torsion-to-interval computation used to make it effective.

Time	Source and task	Checkpoint
40 min	HRW23, Section 7.5, printed pp. 78–80: Theorem 7.17, Lemma 7.18, Corollary 7.19	Explain the exceptional $S(M) = \mathbb{R}P^1$ convention and the singular pegboard limit
75 min	HRW23, Section 7.6, printed pp. 80–82: proof of Theorem 1.14; Figures 62–63	Prove the Rolle step and diagrammatically justify both directions of gluing
45 min	HRW23, Sections 7.4 and 7.7, printed pp. 76–77 and 82–84	Derive the Mayer–Vietoris formula; check the four order-five matrices
40 min	HRW22, Section 5.3, published Lemmas 66, 68, Proposition 67, Theorem 69	Explain how torsion computes both input L-space intervals
40 min	Work Exercises 2, 3, 5, 7 and compare with Figures 62–63	Reconstruct proof steps independently
30 min	Oral questions, source-numbering audit, Day 14 bridge	Defend every hypothesis at the board

Total: 4 h 50 min.

Proof checkpoints

- ☐ State HRW23, Theorem 1.14 with **both boundary-incompressibility assumptions** and interiors.
- ☐ Explain why rank one in every Spin^c summand is equivalent to the L-space condition for a rational homology sphere.
- ☐ Derive the Rolle/mean-value chord-tangency lemma without using a picture.
- ☐ Explain how two **same-**Spin^c intersections give a common tangent slope on both curve sets.
- ☐ Explain why compactness of tangent-slope intervals still yields a **rational** missing slope.
- ☐ Reproduce the three geometric cases in the converse using HRW23 Figures 62–63, and say which are source-dependent.
- ☐ Distinguish a chain generator from a **surviving Floer class** and explain the role of opposite intersection signs.
- ☐ Derive the integral-homology-solid-torus Mayer–Vietoris presentation, and state the extra $t_i o_i$ factors in general.
- ☐ Check $\det H_i = -1$, $|r_i| = 5$ for all four gluing matrices.
- ☐ Explain where the genus-one trefoil classification enters Theorem 7.20.
- ☐ Explain the separation between HRW22 torsion input (Theorem 69) and HRW23 gluing proof (Theorem 1.14).

13. Eight serious exercises

Exercise. Exercise 1 - Spin-c minimal rank Assume Y is a rational homology sphere and that each $\widehat{HF}(Y,\mathfrak{t})$ has Euler characteristic ± 1. Prove that Y is an L-space if and only if every summand is one-dimensional. If a summand has surviving classes of both mod-two parity, what is the sharp smallest *possible* dimension compatible with this Euler characteristic? Explain why a pair of raw generators is insufficient to conclude non-L-space status. **Hint:** Split into even and odd homology dimensions e, o and use $|e - o| = 1$.

Exercise. Exercise 2 - A rigorous common-chord lemma Let $c_0, c_1 : [0,1] \to \mathbb{R}^2$ be C^1 immersions with common distinct endpoints p, q. Prove that each has a tangent parallel to $q - p$. Then explain which ingredient of the HRW23 gluing argument allows one to choose common planar endpoints for two intersection points of **equal closed-manifold Spin^c class**. Why is the statement not yet a proof that the common tangent direction is rational? **Hint:** Differentiate $t \mapsto \det(q - p, c_i(t) - p)$.

Exercise. Exercise 3 - Singular-slope sets and logical contrapositives Write the two contrapositive implications needed for

$$Y \text{ L-space} \iff \mathcal{L}^\circ(M_0) \cup h(\mathcal{L}^\circ(M_1)) = \mathbb{Q}P^1.$$

For each implication identify whether HRW uses: (a) two surviving same-Spin^c intersections; (b) tangent-set compactness; (c) two *opposite-sign* intersections near a peg. Explain why the union condition cannot be replaced with a nonempty intersection of the two L-space intervals. **Hint:** Be explicit about the words "not an L-space" and "not all slopes covered".

Exercise. Exercise 4 - Mayer–Vietoris with the row convention Suppose $H_1(M_i) = \mathbb{Z}\langle u_i \rangle$, the primitive rational longitudes λ_i vanish in homology, and each $\mu_i \mapsto u_i$. Let

$$H = \begin{pmatrix} s & r \\ q & p \end{pmatrix}, \quad sp - rq = -1$$

act on **row** vectors (λ, μ). Derive a complete presentation of $H_1(M_0 \cup_h M_1)$ from Mayer–Vietoris and show it is $\mathbb{Z}/|r|$ for $r \neq 0$. What happens when $r = 0$? Why can $|r|$ not always be used without correction when the pieces have torsion in H_1? **Hint:** Write the two relations for λ_1 and μ_1.

Exercise. Exercise 5 - Four gluing matrices and a fake order-six candidate Check the determinants and first-homology orders of

$$\begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}, \begin{pmatrix} -3 & 5 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} -2 & 5 \\ 1 & -2 \end{pmatrix}.$$

Then analyze $K = \left(\begin{smallmatrix} 1 & 6 \\ 0 & -1 \end{smallmatrix}\right)$. Explain why the existence of an order-six homology gluing does not contradict Theorem 7.20. Which additional property would be needed for a contradiction? **Hint:** Separate $|H_1|$ from the L-space condition.

Exercise. Exercise 6 - Local systems and parity Consider two transverse embedded circles with r, s-dimensional local systems. Explain why a geometric intersection yields rs chain generators. If

an even-parity and an odd-parity intersection survive in the same $\mathfrak{t}$, show that this obstructs the L-space condition. Then explain why the exceptional full-circle convention for multiple components and nontrivial local systems is necessary in HRW23, Theorem 7.17. **Hint:** Review HRW23, Corollary 4.10 and Proposition 6.7.

Exercise. Exercise 7 - Torsion-to-gluing pipeline State HRW22, Theorem 69 in its published numbering and define $D^\tau(M)$, j_*, and the projectivized obstruction set. Suppose you have already determined two *actual* Floer-simple intervals for boundary-incompressible pieces. Explain precisely how to test Theorem 1.14. Construct one pair of open arcs on $\mathbb{R}P^1$ that overlaps but does not cover, and one that covers. Do **not** claim the abstract arcs come from named manifolds. **Hint:** Work modulo one to understand the circle topology.

Exercise. Exercise 8 - The rank-five barrier Explain why Theorem 1.8 gives an order-five **lower bound** for toroidal L-spaces but does not rule out order six. Then reproduce the nontrivial proof outline of HRW23, Theorem 7.20: the $t_0t_1o_0o_1\Delta$ formula, reduction to integral homology solid tori, the matrix coefficient $|r|$, the interval inequalities, and the final genus-one trefoil identification. Identify which parts are complete computations in today's notes and which require reading source Figures/arguments. **Hint:** See printed pp. 82–84.

14. Hints and selected full solutions

Solution 1 - parity forces dimension at least three

Let $e, o \geq 0$ be the dimensions of the even and odd homology in a fixed Spin^c summand. The Euler characteristic condition is $|e - o| = 1$, and the total dimension is $e + o$. If both parities appear, $e, o \geq 1$. The least solution is $(e, o) = (2, 1)$ or $(1, 2)$, giving $e + o = 3$. Thus a summand with two **surviving** opposite-parity classes cannot be one-dimensional (indeed must have dimension at least three). The qualification "surviving" matters because a canceling pair of chain generators may disappear in homology.

Solution 2 - the common chord lemma

For $v = q - p \neq 0$, set $F_i(t) = \det(v, c_i(t) - p)$. Then $F_i(0) = F_i(1) = 0$, so Rolle gives $F_i'(t_i) = \det(v, c_i'(t_i)) = 0$. Since c_i is immersed, $c_i'(t_i) \neq 0$ and is parallel to v. Equality of closed Spin^c gradings of two Floer intersections gives $[c_0 - c_1] = 0 \in H_1(T^2)$, which is the condition allowing common planar lifts. The resulting v can have irrational slope, because p and q need not be lattice points. Rationality is obtained later by the limiting singular pegboard intervals with rational endpoints, not directly from Rolle.

Solution 3 - two contrapositives

Let P be "Y is an L-space" and Q be "the union of interior L-space slopes equals $\mathbb{Q}P^1$". The two useful contrapositives are $\neg P \Rightarrow \neg Q$ (proves $Q \Rightarrow P$) and $\neg Q \Rightarrow \neg P$ (proves $P \Rightarrow Q$). The first uses same-Spin^c intersections, the common-chord lemma, limiting tangent sets and rational endpoints. The second starts with a common singular rational slope and, using Figures 62–63, produces two persistent intersections of opposite parity. A nonempty *intersection* of the two L-space intervals supplies no information about uncovered directions, so it cannot replace the full-union criterion.

Solution 4 - presentation over $\mathbb{Z}$

Mayer–Vietoris gives

$$H_1(Y) = \frac{\mathbb{Z}u_0 \oplus \mathbb{Z}u_1}{\langle ru_0,\ u_1 - pu_0 \rangle} \cong \mathbb{Z}/|r| \quad (r \neq 0).$$

If $r = 0$, then $u_1 = pu_0$ and no nonzero multiple of u_0 is killed, so $H_1(Y) \cong \mathbb{Z}$: the result is not a rational homology sphere. For rational homology solid torus pieces the primitive rational longitude may represent nontrivial torsion and $H_1(M_i)$ may have torsion subgroup of order t_i; the full formula carries factors $t_0t_1o_0o_1$.

Solution 5 - explicit matrices

All four displayed matrices have determinant -1 and second entry of the first row $r = 5$, hence integral-homology-solid-torus gluings have $H_1 \cong \mathbb{Z}/5$. For K, $\det K = -1$ and $r = 6$, hence $H_1 \cong \mathbb{Z}/6$ under the same hypotheses. This does not produce an order-six **L-space**: the theorem rules out that additional Floer-theoretic property for a prime toroidal gluing. An arbitrary integral homology calculation does not imply minimal Heegaard Floer rank.

Solution 6 - local-system ranks

At an intersection of curves with fibers $\mathbb{F}^r$ and $\mathbb{F}^s$, the generator fiber is their tensor product, dimension rs. The Floer differential counts bigons weighted by parallel transport, so some combinations of generators may cancel. If opposite parities survive in the same closed Spin^c class, the Euler characteristic test forces homology dimension at least three. Multiple components or nontrivial local systems can obstruct the minimum-rank filling condition for *every* slope, motivating the $S(M, \mathfrak{s}) = \mathbb{R}P^1$ convention in the definition of singular sets.

Solution 7 - interval topology

Take arcs $I_0 = (0.1, 0.8)$ and $I_1 = (0.6, 1.2)$ modulo 1: their union is the full circle. The arcs $J_0 = (0.1, 0.6)$ and $J_1 = (0.5, 0.9)$ overlap but omit an open arc around zero. In practice, HRW22, Theorem 69 computes each input interior by taking the **largest interval containing a certified slope** and avoiding $\mathbb{P}(j_*^{-1}(D^\tau))$. After transporting the second by h, apply HRW23, Theorem 1.14 to test full coverage, provided both pieces are boundary incompressible.

Solution 8 - where the hard analysis lives

Theorem 1.8 implies $\dim \widehat{HF}(Y) \geq 5$ for a separating essential torus, hence $|H_1(Y)| \geq 5$ when Y is an L-space; order six is not excluded. Theorem 7.20 first writes $|H_1(Y)| = t_0 t_1 o_0 o_1 \Delta$, then uses curve-intersection estimates to force the integral homology solid torus case for orders at most six. With the row-vector gluing matrix, $\Delta = |r|$. Matrix inequalities derived from the necessary interval coverage exclude $r = 1, 2, 3, 4, 6$ and leave four matrices at $|r| = 5$. Finally the interval endpoints force genus one and a separate knot-detection theorem identifies the pieces as trefoil exteriors. Our Mayer–Vietoris and determinant calculations are complete; the $r = 3, 4, 6$ elimination and trefoil classification are retained as source-based proof-sketch steps.

15. Oral qualifying questions with answers

Question. Q1. State the gluing criterion without losing a hypothesis. **Answer.** If M_0, M_1 are compact oriented boundary-incompressible manifolds with torus boundary and $h : \partial M_1 \to \partial M_0$ reverses orientation, then $M_0 \cup_h M_1$ is an L-space iff $\mathcal{L}^\circ(M_0) \cup h(\mathcal{L}^\circ(M_1)) = \mathbb{Q}P^1$. It requires **union of interiors covering all rational slopes**.

Question. Q2. What turns two Floer intersections into a tangent slope? **Answer.** With equal closed Spin^c grading, arcs on the two curves between the intersection points have the same lifted endpoints in the planar cover. Each path has a tangent parallel to the chord connecting these endpoints, by Rolle's theorem applied to a determinant. Taking tight pegboard limits and using rational endpoints of singular intervals gives a rational common tangent direction.

Question. Q3. Why are Figures 62–63 needed in the converse? **Answer.** A common tangent direction by itself does not guarantee two persistent Floer classes, much less opposite parity. HRW's local pegboard case analysis constructs two **minimal-position** intersections with opposite orientation signs, including the subtle case of long horizontal segments. These give classes obstructing the L-space minimum rank.

Question. Q4. Explain the elliptic-involution convention. **Answer.** HRW23's original pairing uses $\bar{h}$, with an elliptic-involution correction. HRW22 proves the manifold-derived immersed-curve invariant is elliptic invariant, so one may write the pairing with h for genuine manifold invariants. One cannot automatically remove it for arbitrary abstract decorated curves.

Question. Q5. Why does the theorem not apply to filling by an ordinary solid torus? **Answer.** Theorem 1.14 assumes both pieces are boundary incompressible, whereas a solid torus is boundary compressible. Dehn-filling slopes are characterized by the tangent-slope theorem (HRW23, Theorem 7.17), and the ordinary solid torus must be treated separately.

Question. Q6. Derive the homology order in the integer case. **Answer.** In row coordinates (λ, μ), $H = \left(\begin{smallmatrix} s & r \\ q & p \end{smallmatrix}\right)$ maps λ_1 to $s\lambda_0 + r\mu_0$ and μ_1 to $q\lambda_0 + p\mu_0$. The longitudes vanish in $H_1(M_i) = \mathbb{Z}$, giving relations $ru_0 = 0$ and $u_1 = pu_0$. Thus $H_1(Y) = \mathbb{Z}/|r|$, where $|r| = \Delta(\lambda_0, h\lambda_1)$.

Question. Q7. Why can a toroidal L-space not have first homology order six? **Answer.** The rank-five lower bound alone does not rule this out. HRW23, Theorem 7.20 uses interval coverage and an orientation-reversing $SL^{\pm}(2, \mathbb{Z})$ matrix enumeration to exclude the order-six case; $|H_1(Y)| = 6$ is possible as a homology computation but not as a prime toroidal **L-space**.

Question. Q8. How do the two HRW papers work together here? **Answer.** HRW23 proves the general geometric gluing theorem using tangent slopes and the bordered pairing. HRW22 proves that

for Floer-simple pieces the interior L-space slope interval can be computed from Turaev torsion. Thus torsion supplies the **input intervals** and the 2023 gluing theorem supplies the **criterion** for the resulting closed manifold.

16. Exact references and Day 14 bridge

[**HRW23**] J. Hanselman, J. Rasmussen, L. Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, arXiv:1604.03466v3 (25 April 2023). Relevant source locations: Theorems **1.2, 1.8, 1.14** (intro, printed pp. 2–6); **Theorem 7.17**, **Lemma 7.18**, **Corollary 7.19** (Section 7.5, printed pp. 78–80); **proof of Theorem 1.14** (Section 7.6, printed pp. 80–82, **Figures 62–63**); **Theorem 7.20** and proof (Section 7.7, printed pp. 82–84). The separate double-torus lower bound and the rank-five trefoil example appear in Section 7.4, including Figure 59.

[**HRW22**] J. Hanselman, J. Rasmussen, L. Watson, *Heegaard Floer homology for manifolds with torus boundary: properties and examples*, *Proc. London Math. Soc.* **125** (2022), 879–967, open-access published full text. Relevant source locations: **Proposition 62** (Section 5.1, signed ray intersections and Turaev torsion); **Lemma 66, Proposition 67, Lemma 68, Theorem 69** (Section 5.3, graph, black-white pegs, torsion difference sets, and interval reconstruction); **Theorem 7** (elliptic-involution symmetry used to simplify the pairing convention). The published numbering differs from the 2018 first arXiv version.

[**LOT18**] R. Lipshitz, P. Ozsvath, D. Thurston, *Bordered Heegaard Floer Homology*, *Memoirs AMS* **254** (2018), no. 1216; supporting background for the box tensor pairing. **Do not** replace the HRW geometric proof with a general statement that the LOT theorem alone implies the L-space gluing criterion.

[**RR17**] J. Rasmussen and S. D. Rasmussen, *Floer Simple Manifolds and L-Space Intervals*, *Advances in Mathematics* **322** (2017), 738–805; supporting external classification of Floer-simple intervals referenced in HRW23, Section 7.7. [**Ghiggini08**] P. Ghiggini, *Knot Floer homology detects genus-one fibred knots*, *Amer. J. Math.* **130** (2008), no. 5, 1151–1169; cited as [14] in HRW23 and used in the final trefoil-identification step.

Next: Day 14. The course turns from L-space gluing to the geometric consequences of essential tori: the rank-five lower bound, *pinches* and degree-one maps (HRW23, Theorems 1.8, 1.12, 7.15), and their relationship with the examples and geometric applications in HRW22. The main task will be to extract the precise geometric reason the Floer rank cannot decrease under the particular pinches treated by HRW, and to separate that theorem from the much broader open degree-one-map question.