HRW Qualifying Examination - Day 14

Pinches, Intersection Monotonicity, and Toroidal Rank Bounds

2026-11-05

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Course date: Thursday, November 5, 2026. **Study time:** 4 hours 50 minutes. **Level:** PhD qualifying examination in geometric topology. **Coefficients:** $\mathbb{F} = \mathbb{F}_2$. **PDF:** *HRW_Day14.pdf*.

Concept links: *Immersed Curves* · *Heegaard Floer Homology* · *Bordered Pairing Theorem* · *Rational Longitude* · *Pegboard Diagrams* · *Heegaard Floer Homology Solid Tori* · *Thurston Norm* · *Mayer-Vietoris* · *Essential Torus*.

> **Warning.** Scope and source discipline We use **HRW23** for the three-author 2023 arXiv v3 paper, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, with its own printed page numbering. **HRW22** is the three-author published *Properties and examples* companion. Theorem 7.15, not Theorem 1.12 alone, is the source of the sharpened rational-homology-sphere inequality. A **source theorem** and a **derived lemma** are distinguished throughout. The hardest pegboard-counting steps are labeled **incomplete proof sketch**: Figures 54–58, not a generic isotopy argument, supply the missing geometric details.

1. Theorem map and learning goals

Today's problem is a quantitative one: **when an essential torus is pinched, why does the rank of $\widehat{HF}$ not increase?** The geometric answer is that making one of the curve sets straight and forgetting its puncture windings cannot create additional unavoidable intersections with the other set. The word *unavoidable* matters: Floer homology is not the raw intersection count of an arbitrary drawing.

> **Theorem.** Source: HRW23, Theorem 1.12 (p. 6) If Y is a **toroidal integer homology sphere** and Y_0 results from a pinch along a torus, then
>
> $$\dim_{\mathbb{F}} \widehat{HF}(Y) \;\geq\; \dim_{\mathbb{F}} \widehat{HF}(Y_0).$$

> **Theorem.** Source: HRW23, Theorem 7.15 (printed p. 71) Suppose $Y = M_0 \cup_h M_1$ is a rational homology sphere and M_i are rational homology solid tori. Let λ_1 be the **primitive rational longitude** of M_1, let $Y_0 = M_0(h(\lambda_1))$, let n be the order of $i_{1*}([\lambda_1]) \in H_1(M_1)$, and put
>
> $$m = |\operatorname{Spin}^c(M_1)| = |H_1(M_1, \partial M_1)|.$$

Then

$$\boxed{\dim \widehat{HF}(Y) \geq nm \dim \widehat{HF}(Y_0).}$$

Theorem. Source: HRW23, Theorem 1.8 (p. 5), proved in Section 7.4 (printed pp. 74–77) If a closed oriented three-manifold contains a **separating essential torus**, then

$$\boxed{\dim_{\mathbb{F}} \widehat{HF}(Y) \geq 5.}$$

The rank-five bound is **not** a formal consequence of the pinch inequality: they use related pegboard geometry but are separate arguments. Today we prove their algebraic inputs, explain precisely the geometric reductions, compute the multiplicity factors, and isolate the diagram-dependent portions. At the end, you should be able to reconstruct the entire *dependency graph* at the board.

2. Pinching and the longitude: exact topological setup

Let M be a compact connected oriented three-manifold with a single torus boundary and $H_*(M;\mathbb{Q}) \cong H_*(S^1;\mathbb{Q})$. The inclusion

$$j_* : H_1(\partial M;\mathbb{Z}) \longrightarrow H_1(M;\mathbb{Z})$$

has rank-one image. Its kernel is generated by a nonzero class called the **homological longitude** ℓ. Write

$$\boxed{\ell = n\lambda}$$

with λ primitive in $H_1(\partial M;\mathbb{Z})$ and $n \geq 1$. This λ is the **rational longitude**, and its image in $H_1(M)$ has order exactly n. The distinction $\lambda \neq \ell$ when $n > 1$ is crucial in the rational theorem. The boundary slope of λ is defined independently of its orientation.

Choose an orientation-reversing identification $h : \partial M_1 \to \partial M_0$ and suppose

$$Y = M_0 \cup_h M_1$$

is a rational homology sphere. The **pinched filling** is

$$\boxed{Y_0 = M_0(h(\lambda_1)).}$$

Equivalently, replace M_1 by a genuine solid torus whose meridian is attached along $h(\lambda_1)$. It is *not* the filling along the meridian of M_1 and not necessarily the filling along the homological longitude $n\lambda_1$: an integral multiple does not define a new primitive slope.

For integer homology solid tori (in particular, knot exteriors in integer homology spheres), $n = 1$ and there is a canonical topological **degree-one pinch map** $Y \to Y_0$, obtained by replacing the second piece with a solid torus compatibly with its longitude. In the general rational case, HRW write that a **degree-n map exists in certain settings**. The numerical inequality of Theorem 7.15 does **not** require proving that such a degree-n map exists for every rational pair. This nuance matters for the final degree-one-map question.

Warning. Three different operations **Dehn filling:** choose any primitive slope α on ∂M_0 and attach a solid torus. **Pinch:** the special filling $\alpha = h(\lambda_1)$ associated to an existing torus decomposition. **Arbitrary degree-one map:** a continuous map $f : Y \to Z$ with $f_*[Y] = [Z]$. Only the first two are present in the HRW argument; the broad rank-monotonicity question for arbitrary degree-one maps is not settled by Theorem 7.15.

3. Homology and the integers n and m

Since $H_1(M_1, \partial M_1)$ is finite, the set $\mathrm{Spin}^c(M_1)$ is a finite torsor over $H^2(M_1) \cong H_1(M_1, \partial M_1)$, by Poincare–Lefschetz duality. Thus m counts the absolute Spin^c structures of the second piece. The number n measures something different: torsion of the *image of the rational longitude* in $H_1(M_1)$.

3.1 A complete homology computation

By the long exact sequence of the pair,

$$H_1(\partial M_1) \xrightarrow{j_*} H_1(M_1) \longrightarrow H_1(M_1, \partial M_1) \longrightarrow H_0(\partial M_1) \xrightarrow{\cong} H_0(M_1).$$

The last arrow is an isomorphism because M_1 and ∂M_1 are connected. Exactness gives the derived identity

$$\boxed{H_1(M_1, \partial M_1) \cong \operatorname{coker} j_*}. \tag{3.1}$$

Thus $m = |\operatorname{coker} j_*|$ whenever this group is finite.

Derived algebraic model (not claimed as a specified manifold). Suppose

$$H_1(M_1) = \mathbb{Z} \oplus \mathbb{Z}/2, \qquad j_*(a\mu + b\lambda) = (a, [b]_2).$$

Then $j_*(\lambda) = (0, [1]_2)$ has order $n = 2$, while j_* is surjective, so $m = 1$. The homological longitude is 2λ. We would obtain $nm = 2$, not 1, in Theorem 7.15. This example exposes the danger of confusing the two integers.

3.2 The integer homology-sphere specialization

If Y is an integer homology sphere cut along a separating torus, Mayer–Vietoris implies that both M_i are integer homology solid tori: $H_1(M_i) \cong \mathbb{Z}$, with primitive null-homologous longitudes. In particular,

$$n = 1, \qquad m = 1,$$

and Theorem 7.15 reduces to Theorem 1.12. As a useful separate computation, if each $H_1(M_i) = \mathbb{Z}\langle u_i \rangle$, and the gluing sends $\lambda_1 \mapsto s\lambda_0 + r\mu_0$, $\mu_1 \mapsto q\lambda_0 + p\mu_0$, the Mayer–Vietoris relations become

$$ru_0 = 0, \qquad u_1 = pu_0.$$

Therefore $H_1(Y) \cong \mathbb{Z}/|r|$ if $r \neq 0$. For an integer homology sphere $|r| = 1$, so $h(\lambda_1)$ is a slope at distance one from λ_0. The general rational theorem still requires its explicit factor nm.

4. Geometric pairings and the correct intersection inequality

Let $\Gamma_i = \mathcal{C}(M_i) = \widehat{HF}(M_i)$ denote HRW's decorated immersed-curve **invariant**, not a vector space. Transport the second curve collection by the gluing map and the relevant elliptic correction. The latter can be suppressed for actual manifold invariants using HRW22's elliptic symmetry (Theorem 7 in the published companion). The geometric pairing theorem, HRW23 Theorem 1.2, gives

$$\widehat{HF}(Y) \cong HF(\Gamma_0, h\Gamma_1).$$

For clarity, denote the second curve set after transport by Γ_1 again. The minimum geometric intersection number with rank multiplicities is $i(\Gamma_0, \Gamma_1)$; local systems multiply the number of Floer generators at a geometric intersection.

Theorem. Source-dependent inequality used in HRW23, proof of Theorem 7.15

$$\boxed{\dim HF(\Gamma_0, \Gamma_1) \geq i(\Gamma_0, \Gamma_1).}$$

For noncommensurable primitive components the dimension is **equal** to the minimal intersection count times the local-system ranks (HRW23, Corollary 4.10). For commensurable components, additional self-pairing-type Floer homology may remain even though the minimum geometric intersection is zero. Equality is **not** asserted without the extra hypotheses.

Let $L = L_{p,q}$ be a simple closed, **unoriented slope curve** in T_{M_0} representing $h(\lambda_1)$. Because Y is a rational homology sphere, $h(\lambda_1)$ is not the rational longitude of M_0. By HRW's homological restriction on manifold curve components (Corollary 6.6), no component of Γ_0 is commensurable with L. Thus the filling pairing gives

$$\boxed{\dim \widehat{HF}(Y_0) = i(\Gamma_0, L).} \tag{4.1}$$

Combining the two formulas, the entire pinch inequality is reduced to the *purely geometric* comparison

$$i(\Gamma_0, \Gamma_1) \geq nm\, i(\Gamma_0, L). \tag{4.2}$$

This is why Section 7.3 devotes several pages to moving corners past the puncture.

5. Loose representatives and the factor nm

HRW call an embedded closed slope line **loose** when it can be displaced to nearby parallel geodesics without interacting with the puncture. Construct $\hat{\Gamma}_1$, the **loose representative**, by forgetting the puncture while retaining the underlying torus homology classes and ranks of local systems. The source gives the following explicit sequence of operations: (i) replace nontrivial local systems by trivial systems of the same rank; (ii) resolve self-intersections to split off closed components; (iii) discard components contractible in the filled torus; (iv) tighten the surviving components in a flat torus; (v) pass corners across the peg. See HRW23, Figures 54–55.

For each $\mathfrak{s} \in \mathrm{Spin}^c(M_1)$, Corollary 6.6 tells us that the **oriented sum** of components in the corresponding lift projects to the homological longitude $n\lambda_1$ (with the appropriate homology orientation). Consequently, its loose representative contains **at least** n **units of longitudinal slope multiplicity**, counted with local-system rank. There are m absolute Spin^c structures, so after transporting the curves to ∂M_0,

$$\boxed{\hat{\Gamma}_1 \text{ contains at least } nm \text{ copies of the slope } L}$$

for purposes of geometric intersection counting (with rank weights). Some extra parallel components could appear in canceling oriented pairs; that only increases the *unsigned* intersection number.

> **Proof.** Derived linear-algebra check of the multiplicity lower bound Fix a primitive homology class $v \in \mathbb{Z}^2$. Suppose the loose multicurve contains oriented slope copies $\varepsilon_j k_j v$, where $\varepsilon_j = \pm 1$ and $k_j \in \mathbb{Z}_{>0}$ includes local-system rank and covering multiplicity. If the total class is nv, then $\sum_j \varepsilon_j k_j = n$. Therefore $\sum_j k_j \geq |\sum_j \varepsilon_j k_j| = n$. Repeating over m Spin^c summands proves the factor nm. No assumption that all components are coherently oriented is needed.

Since geometric intersection with a fixed nonparallel collection is additive over slope copies,

$$\boxed{i(\Gamma_0, \hat{\Gamma}_1) \geq nm\, i(\Gamma_0, L) = nm \dim \widehat{HF}(Y_0).} \tag{5.1}$$

The difficult remaining comparison is

$$\boxed{i(\Gamma_0, \Gamma_1) \geq i(\Gamma_0, \hat{\Gamma}_1).} \tag{5.2}$$

We now discuss precisely why passing through a peg need not make the minimal intersection count larger.

6. Five elementary modifications and the corner cost

Fix a sufficiently small $\epsilon > 0$ and take minimal-position ϵ-pegboard representatives of the curve sets. In the square fundamental domain of $T^2 \setminus \{z\}$, the puncture is surrounded by a disk of radius ϵ. A corner of Γ_1 is an arc of a circle near that disk joining two straight segments. A **corner slide** replaces the corner by an arc on the *opposite* side of the puncture, near the circle of radius 2ϵ (HRW23 Figure 54). This move is not necessarily a homotopy in the punctured torus; it is allowed only in constructing the loose representative in the **filled** torus.

Moves (i) and (ii) do not alter the relevant geometric intersection count; deleting a component clearly cannot increase it. Tightening in the punctured torus does not increase the **minimal** count. The subtle move is passing a corner across the peg, which may increase local intersections with some corners of Γ_0 while decreasing them with others.

6.1 The four corner types (HRW23, Figure 55)

Fix a corner c_1 of Γ_1. The arc c_1 cuts the square tile into an **inside** containing the puncture and an outside. Extend each corner c_0 of Γ_0 to a strand entering and leaving the square tile. HRW classify the strand endpoints relative to the inside:

(a) both endpoints lie inside; the original configuration contributes zero intersections near the corner, but the slide creates **two**. **(b)** exactly one endpoint lies inside; there is **one** intersection before and after. **(c)** both lie outside and the c_0 corner turns in the same direction without wrapping; there are **zero** intersections before and after. **(d)** both lie outside and c_0 wraps or turns the other way; there are **two** before, **zero** after.

Write N_a, N_b, N_c, N_d for the four numbers of corners of Γ_0. Then the **change** in the number of local intersections under the slide is exactly

$$\boxed{\Delta I = 2N_a - 2N_d.} \tag{6.1}$$

This arithmetic statement follows from the four local pictures, once those pictures are established. It is the key quantitative reduction of the proof.

> **Warning.** Dangerous shortcut Saying "move the corner over the puncture; intersections decrease'' is false without hypotheses. A **single** type (a) corner raises the intersection count by two. One must prove the global inequality $N_d \geq N_a$, or treat the special straight-corner case separately.

7. Why there are enough good corners

The source distinguishes a **true corner** (the singular pegboard representative changes direction) from a **π-corner** (the singular representative is straight, although the ϵ-representative touches a peg). The local count (6.1) is useful for a true corner.

7.1 Cyclic charging argument: incomplete geometric proof sketch

Fix the true corner c_1. Let $s_{\text{in}}, s_{\text{out}}$ be the boundary points where its arc enters and leaves the square tile. In the singular pegboard model, the *inside* occupies **strictly less than half** the perimeter of the tile. If the corresponding singular angle is $\phi > \pi$, the short side cut from the boundary is less than half. For sufficiently small ϵ, the inequality persists after thickening by $O(\epsilon)$; HRW explicitly control endpoints by constants proportional to ϵ (Figure 56).

Traverse the closed curve Γ_0 through its cyclic list of corners $c_0^1, \dots, c_0^k$. If a corner is type (a), the outgoing boundary endpoint is inside the short region. The next incoming endpoint is approximately **antipodal** to this outgoing endpoint: between puncture corners the curve has not changed its direction in the singular diagram. Since the inside occupies less than half the perimeter, the next incoming endpoint must lie *outside* the inside. Subsequent corners can only be of types (b), (c), or (d); any repeated type (c) corners are straight in the singular diagram and preserve the outside condition. Before the cyclic sequence reaches another type (a), it must pass a type (d). Thus each type (a) can be **charged to a distinct later type (d)**.

> **Proof.** Derived combinatorial core Suppose a finite cyclic word in symbols a, b, c, d has the property: after every a, and before the next a, there is at least one d. Partition the word at its occurrences of a. Each resulting arc contains a d, and these arcs are disjoint except at endpoints. Therefore the number of d symbols is at least the number of a symbols. Applying this to the corner-type word yields $N_d \geq N_a$, hence $\Delta I \leq 0$ by (6.1).

The **nontrivial geometric input** is the stated property of the corner-type word, which depends on antipodal endpoints and the strict half-perimeter inequality. This is verified in HRW23 printed pp. 73–74, Figures 55–57. Thus for a **true corner** passing it across the peg does not increase the total minimal intersection count.

8. Straight corners and HF solid tori

If c_1 is a π-corner, the singular representative is straight. Its inside region may occupy **more than half** of the tile, invalidating the charging argument. Indeed, sliding **one** such corner can increase intersections. HRW do not pretend that the same proof works.

Instead, slide all *true* corners first, and use the other allowed moves to reduce to a curve whose remaining corners are all straight: a **Heegaard Floer homology solid-torus curve**. It lies in a small annular neighborhood of a primitive straight slope when the punctures are filled. For such a curve, take an annular neighborhood containing the 2ϵ circle around the peg. Any segment of Γ_0 entirely inside this annulus can be ignored for the comparison with its loose representative. Any segment that leaves the annulus meets the original and loose representatives **the same number of times** (HRW23 Figure 58). Thus the final straightening also cannot increase the count.

> **Warning.** What is actually preserved? The sequence $\Gamma_1 \rightsquigarrow \hat{\Gamma}_1$ is **not asserted to preserve the type D object**. It is a comparison of intersection numbers after forgetting puncture windings, and the local-system monodromy may be replaced by trivial monodromy. One does **not** conclude $HF(\Gamma_0, \Gamma_1) \cong HF(\Gamma_0, \hat{\Gamma}_1)$. Only the rank **inequality** is needed.

The two cases together prove (5.2). Their geometric verification is the proof of HRW23 Theorem 7.15, printed pp. 72–74.

9. Proof assembly: HRW23 Theorem 7.15

We can now expose the logical chain without suppressing the main inequality. Let $Y = M_0 \cup_h M_1$ be a rational homology sphere, let $Y_0 = M_0(h(\lambda_1))$, and let n, m be as in the theorem. HRW's curve pairing and minimal-position theory give

$$\dim \widehat{HF}(Y) = \dim HF(\Gamma_0, \Gamma_1) \geq i(\Gamma_0, \Gamma_1).$$

The corner-slide analysis gives

$$i(\Gamma_0, \Gamma_1) \geq i(\Gamma_0, \hat{\Gamma}_1).$$

Longitudinal homology in each Spin^c summand gives

$$i(\Gamma_0, \hat{\Gamma}_1) \geq nm\, i(\Gamma_0, L).$$

Finally, the non-longitudinal filling slope ensures that the filling pairing is noncommensurable, so

$$i(\Gamma_0, L) = \dim \widehat{HF}(Y_0).$$

Combining these **four** statements,

$$\boxed{\dim \widehat{HF}(Y) \geq nm \dim \widehat{HF}(Y_0).}$$

This is the rational pinch inequality. In the integer homology-sphere case, $n = m = 1$, proving Theorem 1.12.

Proof. Proof dependency audit **Algebraically proved here:** $m = |\operatorname{coker} j_*|$; multiplicity $\sum k_j \geq |\sum \varepsilon_j k_j|$; the four-step rank chain once the source's intersection lemmas are granted. **Source-level substantial proof sketch:** the ϵ-pegboard charging argument and the straight-π-corner annulus analysis. **Previously proved source theorems:** bordered pairing (HRW23 Theorem 1.2), noncommensurable Floer dimensions (Corollary 4.10), and longitude homology (Corollary 6.6). Theorem 7.15 is not claimed to have a fully independent figure-free proof here.

10. Worked algebra and immersed-curve computations

10.1 An integer-homology-sphere splice

Let K_R, K_L be the right- and left-handed trefoils and M_R, M_L their exteriors. Splicing identifies a meridian of one with the longitude of the other and vice versa (with an orientation-reversing sign). These are integer homology solid tori; the splice is an integer homology sphere. The resulting pinched manifold on either side is meridional filling of a trefoil exterior, namely S^3, so

$$\dim \widehat{HF}(Y_0) = \dim \widehat{HF}(S^3) = 1.$$

Theorem 1.12 therefore implies $\dim \widehat{HF}(Y) \geq 1$, a weak but valid bound. The full immersed-curve calculation in HRW23 Figure 3 gives two informative sharper examples: the splice of a right and a left trefoil has rank **9**, while the splice of two right trefoils has rank **7**. These ranks arise from intersection Floer homology, not from the pinch inequality alone.

10.2 Determinants do not prove the pinch theorem

Let primitive slope vectors be $u = (2, 3)$ and $v = (1, 4)$. Straight representatives in a torus meet

$$\Delta(u, v) = |\det(u, v)| = |8 - 3| = 5$$

times. If they carry local systems of ranks $r = 2$ and $s = 3$, their intersection chain complex has $5rs = 30$ generators, and for noncommensurable geodesics the differential is zero: $\dim HF = 30$ (HRW23 Corollary 4.10). But an arbitrary Γ_1 in Theorem 7.15 is not merely a primitive straight line; its corners can create unavoidable extra intersections, and its local system can be nontrivial. The slope determinant is only the **loose baseline**.

10.3 Explicit rank accounting for a slide

Suppose near one corner of Γ_1 the other curve has $N_a = 2$, $N_b = 3$, $N_c = 1$, $N_d = 4$. Before sliding, the local contributions are $0 + 3 + 0 + 8 = 11$. After sliding they are $4 + 3 + 0 + 0 = 7$. The net change is

$$\Delta I = 7 - 11 = -4 = 2(2 - 4).$$

This is a **local combinatorial example**, not a claimed complete train-track configuration. It shows numerically why comparing the type (a) and type (d) counts is the correct complexity measure.

10.4 Artificial rational multiplicity example

Return to the algebraic model $H_1(M_1) = \mathbb{Z} \oplus \mathbb{Z}/2$ from Section 3. It has $n = 2$, $m = 1$. If a hypothetical compatible gluing has a filling with $\dim \widehat{HF}(Y_0) = 5$, the rational theorem would give

$$\dim \widehat{HF}(Y) \geq 2 \cdot 1 \cdot 5 = 10.$$

If instead $n = 1$ and $m = 3$, the same filling rank would yield 15. These are **conditional numerical consequences**, not constructions of three-manifolds with those exact numerical inputs.

10.5 An elementary parity check

A finite-dimensional chain complex (C, d) over $\mathbb{F}$ satisfies $\dim H(C) = \dim C - 2 \operatorname{rank} d$. Indeed, $\dim \ker d = \dim C - \operatorname{rank} d$ and $\operatorname{im} d \subset \ker d$. Thus adding two intersection generators connected by an acyclic bigon changes chain rank but **not** homology rank. The proof of Theorem 7.15 uses *minimal geometric intersection numbers* rather than raw counts in one drawing precisely to avoid this trap.

11. The essential-torus rank-five barrier

Theorem. Source: HRW23, Theorem 1.8 For a closed oriented Y containing a **separating essential torus**,

$$\dim \widehat{HF}(Y) \geq 5.$$

This is not proved by simply noting that two distinct essential curves have five intersections. For example, two straight slopes at distance one intersect once. A boundary-incompressible manifold invariant is more constrained than an arbitrary straight curve. HRW23 Section 7.4 proves the bound by **sorting the types of pegboard corners** and invoking topological nonvanishing to exclude false low-rank pictures.

11.1 Source corner estimate

Suppose Γ_0 is *not* a Heegaard Floer homology solid-torus curve. If Γ_1 has T true corners, P straight π-corners, and L loose components, HRW prove the geometric lower bound (equation (5), printed p. 75):

$$\boxed{i(\Gamma_0, \Gamma_1) \geq 2T + P + L.} \tag{11.1}$$

Each true corner of the second curve forces two intersections with the non-solid-torus first curve; each π-corner forces one; a loose component forces at least one. A closed non-straight curve generally needs **at least two true corners**, because if its direction changes once it must change back before closing. The source explains the potential full-wrap exception and rules it out for actual manifold invariants using Corollary 7.19.

Therefore, when neither piece is HF-solid-torus-like, (11.1) gives at least four intersections. To exclude **exactly four**, HRW examine the two-segment singular pegboard possibilities. One possible arrangement requires an immersed annulus, so an admissible perturbation yields Floer rank six; another forces a fifth geometric intersection. This is worked through in HRW23 Figure 59, printed pp. 75–76. The case where one or both curve sets are HF-solid-torus-like is handled separately (Figure 60): topology forces **both tight and suitable loose components**, and their Floer pairings raise the total rank to at least five.

Proof status: The estimate (11.1) and the classification of equality are **source theorems/figure-dependent proof sketch**, not a universal claim about arbitrary pairs of curves. A topological input is nonvanishing of Heegaard Floer homology in certain torsion Spin^c summands, used to ensure at least one relevant curve for each longitude-torsion class (HRW23 printed p. 74). This is stronger than the elementary determinant estimate.

11.2 Corollaries with complete short proofs

Theorem. Source: HRW23, Corollaries 1.9 and 1.10 A prime rational homology sphere with $\dim \widehat{HF}(Y) < 5$ is geometric; an integer homology-sphere L-space is atoroidal.

Derivation. A torus in a rational homology sphere is separating (its homology class vanishes over $\mathbb{Q}$). If it were essential, Theorem 1.8 would force rank at least five. Therefore a rank-less-than-five rational homology sphere is atoroidal. The geometric conclusion uses geometrization (and the standard irreducibility/geometric decomposition considerations for a **prime** manifold). For an integer homology-sphere L-space, $\dim \widehat{HF}(Y) = |H_1(Y)| = 1 < 5$, so no essential separating torus exists. Every torus in an integer homology sphere separates, so the manifold is atoroidal. The atoroidality conclusion here is immediate once Theorem 1.8 is known.

Warning. Rank five is genuinely sharp HRW23 Figure 59 depicts a toroidal L-space rank-five realization (see Theorem 7.20). Do not claim $\dim \widehat{HF} > 5$ for every essential separating torus, or that Theorem 7.15 alone supplies the rank-five bound.

12. The 2022 companion: solid-torus curves and norm detection

The two HRW papers have different roles today. The 2023 main paper proves the pinching and rank bounds. The 2022 companion, **Section 1.5**, spells out the exceptional HF-solid-torus condition in equivalent algebraic, filling, and curve-theoretic forms; **Section 5.2** explains why an over-simplified loose configuration is constrained by the Thurston norm.

Theorem. Source: HRW22, Theorem 27 (Section 1.5) Under the hypotheses that M is compact, connected, orientable, **irreducible**, and a knot complement in an L-space, the following are equivalent: (i) its bordered invariant is invariant under a Dehn twist along the rational longitude (the HF-homology-solid-torus property); (ii) every non-longitudinal filling slope is an L-space; (iii) the immersed curves have **trivial local systems** and can be moved by regular homotopy into a small neighborhood of the rational longitude.

Do not omit the hypotheses, or extrapolate (iii) to every abstract extendable type D module. In particular, the twisted I-bundle over the Klein bottle is an example of an HF-homology-solid-torus type manifold with **incompressible boundary**. It is **not** an ordinary solid torus, and the two notions must be distinguished.

The companion's **Proposition 65, Section 5.2** recovers $-\chi(\Sigma)$ for a norm-minimizing surface from the highest obstructed puncture in a Spin^c grading cover. The construction relies on pairing immersed curves with proper slope arcs to compute knot Floer homology of filling cores, then applying knot Floer norm detection. HRW23 **Proposition 7.13**, proved immediately before the pinch argument, uses a similar doubling/norm comparison to show:

> **Theorem.** Source: HRW23, Proposition 7.13 (printed p. 70) If **the entire manifold curve invariant is loose**, then the manifold is boundary compressible.

This is crucial for the rank-five theorem: on each incompressible side there must be some non-loose information. A single loose component is **not** forbidden, even for an incompressible manifold; the source's point is that the *whole* curve collection cannot be loose. Compare HRW22's discussion of tight and loose curves in a longitude neighborhood.

12.1 Complete algebraic self-pairing calculation

Let $A \in GL_r(\mathbb{F}_2)$ be a local-system monodromy on a primitive embedded circle. When pairing against the **same** decorated circle, the second fiber is dualized; the relevant operator is

$$\Phi = I \otimes I + A \otimes (A^{-1})^T.$$

Under $V \otimes V^* \cong \mathrm{End}(V)$, the kernel is isomorphic to

$$\{X \in \mathrm{End}(V) : AX = XA\}.$$

Indeed, the linear equation $X + AXA^{-1} = 0$ is equivalent to $AX = XA$ over $\mathbb{F}_2$. The identity endomorphism commutes with A, so $\ker \Phi \neq 0$. Thus the self-pairing of this decorated curve is nonzero. This calculation is the **derived algebraic mechanism** behind the use of HRW23 Corollary 4.10 in the doubling argument of Proposition 7.13. It also prevents a common mistake: a tight-versus-loose or loose-versus-loose pair can contribute Floer homology even when its minimal geometric intersection number is zero.

13. Degree-one maps: what is and is not proved

HRW23 pose **Question 1.13** immediately after Theorem 1.12: if $f : Y \to Y_0$ is *any* degree-one map between closed connected oriented three-manifolds, must

$$\dim \widehat{HF}(Y) \geq \dim \widehat{HF}(Y_0)?$$

The pinch result proves this for the specified torus-cutting operation on integer homology spheres, not for arbitrary maps. There are other rank inequalities in subsequent literature, but none may be silently substituted for a general theorem answering this question. Even the rational factor nm from Theorem 7.15 should not be called the degree of a guaranteed map for all rational homology solid tori; HRW explicitly qualify the existence of a degree-n pinch map.

The correct oral-exam answer is therefore three layers: **(a)** the proved integer pinch inequality; **(b)** its stronger rational numerical version; **(c)** the broader arbitrary-degree-one question, which is **not proved by either statement**.

14. Exercises - eight exam-level problems

> **Exercise.** Exercise 1: identify n and m (25 min) Prove (3.1) from the pair long exact sequence. For the algebraic homology model $H_1(M) = \mathbb{Z} \oplus \mathbb{Z}/6$ with $j_*(a\mu + b\lambda) = (2a, [2b]_6)$, compute n, m, and the homological longitude. What coefficient nm would appear **conditionally** in Theorem 7.15? **Hint:** Work out the image subgroup and the order of $[2]_6$.
>
> **Exercise.** Exercise 2: a full Mayer–Vietoris calculation (25 min) Let M_i be integral homology solid tori, with bases (λ_i, μ_i) and gluing matrix $H = \left(\begin{smallmatrix} s & r \\ q & p \end{smallmatrix}\right)$, determinant -1. Derive the abelian group $H_1(M_0 \cup_h M_1)$ and identify which value of r gives an integer homology sphere. Why does the rational formula need extra factors? **Hint:** Present the quotient on u_0, u_1.
>
> **Exercise.** Exercise 3: the four-corner arithmetic (20 min) Derive $\Delta I = 2(N_a - N_d)$ using the local configurations of HRW23 Figure 55. Give examples of counts that yield $\Delta I = +2, 0$, and -4. Explain why a single slide can **increase** intersections and why the global charging lemma is needed. **Hint:** Use the before/after counts $(0, 2)$ for (a), $(1, 1)$ for (b), $(0, 0)$ for (c), $(2, 0)$ for (d).

Exercise. Exercise 4: cyclic charging (20 min) Prove that a cyclic word in a, b, c, d in which every a is followed by a d before the next a has $\#d \geq \#a$. Then read HRW23 Figures 56–57 and explain *geometrically* why a true corner forces this condition; identify the exact point where the argument fails at a π-corner. **Hint:** Cut the word at its a entries.

Exercise. Exercise 5: derive Theorem 7.15 (25 min) Write out the four inequalities from $\dim HF(\Gamma_0, \Gamma_1)$ to $nm \dim \widehat{HF}(Y_0)$, identifying the distinct theorem used in each. Why can one not assert $HF(\Gamma_0, \Gamma_1) \cong HF(\Gamma_0, \hat{\Gamma}_1)$? **Hint:** The construction of the loose representative forgets puncture windings and can replace nontrivial local systems.

Exercise. Exercise 6: local systems and paired slopes (20 min) Let $u = (1, 0)$ and $v = (3, 7)$ represent primitive slopes. Equip them with local systems of ranks 2 and 3. Compute the dimension of intersection Floer homology in transverse geodesic minimal position. Then, for a primitive circle with monodromy $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, determine the dimension of its **self**-Floer homology using the centralizer calculation. **Hint:** $\chi_A(t) = t^2 + t + 1$ is irreducible.

Exercise. Exercise 7: rank five and the exceptional case (25 min) State HRW23 Theorem 1.8 with its precise hypothesis. Use equation (11.1) to show why two true corners of Γ_1 force at least four intersections when Γ_0 is not HF-solid-torus-like. Explain why this does **not yet** prove five, and locate the annulus and fifth-intersection arguments in HRW23 Figure 59. **Hint:** Consult printed pp. 75–76.

Exercise. Exercise 8: theorem versus open question (15 min) Formulate (i) Theorem 1.12, (ii) Theorem 7.15, and (iii) Question 1.13 without conflating their hypotheses. Determine which source theorem, if any, justifies a rank inequality for an arbitrary degree-one map $Y \to Z$. Explain whether a degree-n pinch map exists automatically in the rational setting. **Hint:** Read the four sentences preceding Theorem 7.15.

15. Hints and selected solutions

15.1 Exercise 1: torsion arithmetic

Here $j_*(\lambda) = (0, [2]_6)$ has order 3, so $n = 3$ and $\ell = 3\lambda$. The free image of j_* is $2\mathbb{Z}$ (index 2 in $\mathbb{Z}$); the torsion image is $\langle [2]_6 \rangle \cong \mathbb{Z}/3$ (index 2 in $\mathbb{Z}/6$). Thus

$$\operatorname{coker} j_* \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2, \qquad m = 4.$$

The product is $nm = 12$. This is a **calculation in an abstract map of abelian groups**; no realization of these exact input groups by a specific three-manifold is claimed.

15.2 Exercise 2: homology of the splice

$h(\lambda_1) = s\lambda_0 + r\mu_0$ and $h(\mu_1) = q\lambda_0 + p\mu_0$. Both λ_i vanish in their respective $H_1(M_i)$, so Mayer–Vietoris imposes $ru_0 = 0$ and $u_1 = pu_0$. Therefore $H_1(Y) \cong \mathbb{Z}/|r|$ for $r \neq 0$. In particular, $|r| = 1$ gives an integer homology sphere. For rational homology solid tori one must retain the torsion of $H_1(M_i)$, possible nonprimitive homological longitudes, and the relative Spin^c multiplicity; hence the simple determinant calculation cannot substitute for the factor nm.

15.3 Exercise 3: slide accounting

The per-corner changes are $+2, 0, 0, -2$, so $\Delta I = 2N_a - 2N_d$. For $(N_a, N_d) = (2, 1)$ the change is $+2$; for $(3, 3)$ it is 0; for $(1, 3)$ it is -4. The first case proves why a local slide alone is not monotone. The geometric cyclic count $N_d \geq N_a$ for a **true** corner is the essential extra ingredient.

15.4 Exercise 6: intersection and centralizer

$\Delta((1, 0), (3, 7)) = 7$, so the noncommensurable geodesic pairing has dimension $7 \cdot 2 \cdot 3 = 42$. For $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ the centralizer of A is $\mathbb{F}_2[A]$, a two-dimensional vector space because the minimal polynomial is irreducible of degree two. For an embedded primitive circle, HRW23 Corollary 4.10 gives self-pairing dimension $2 \dim \operatorname{Cent}(A) = 4$. A nontrivial local system changes the answer even if the underlying curves agree.

15.5 Exercise 5: the four links

The required chain is

$$\begin{aligned}\dim \widehat{HF}(Y) &= \dim HF(\Gamma_0, \Gamma_1)\\ &\geq i(\Gamma_0, \Gamma_1)\\ &\geq i(\Gamma_0, \hat{\Gamma}_1)\\ &\geq nm\, i(\Gamma_0, L)\\ &= nm\, \dim \widehat{HF}(Y_0).\end{aligned}$$

The middle comparison comes from the corner slides; the next from longitude homology and m summands. No isomorphism involving the **loose representative** has been established or is required.

16. Oral exam: model answers

Question. Q1. State the rational pinch theorem, including both multiplicities. **Model answer.** For a rational homology sphere $Y = M_0 \cup_h M_1$ cut into rational homology solid tori, let λ_1 be the primitive rational longitude and $Y_0 = M_0(h(\lambda_1))$. Let n be the order of the image of λ_1 in $H_1(M_1)$ and $m = |H_1(M_1, \partial M_1)| = |\operatorname{Spin}^c(M_1)|$. Then $\dim \widehat{HF}(Y) \geq nm \dim \widehat{HF}(Y_0)$. The integer homology-sphere pinch has $n = m = 1$.

Question. Q2. Why does the homological longitude matter in the count? **Model answer.** The primitive rational longitude can be torsion of order n in $H_1(M_1)$. The homological longitude is $n\lambda_1$, and Corollary 6.6 says that each Spin^c summand of the oriented immersed-curve invariant projects to this homology class. The loose representative therefore has at least n units of slope multiplicity per summand, yielding nm in total.

Question. Q3. Give the four-step proof of the inequality. **Model answer.** Use the immersed-curve pairing to bound $\dim \widehat{HF}(Y)$ below by $i(\Gamma_0, \Gamma_1)$. Use corner slides to replace Γ_1 by its loose representative without increasing intersections. Use longitudinal homology in m Spin^c summands to bound the loose intersection below by nm times intersection with the straight longitude slope. Finally identify that straight-slope intersection with the Heegaard Floer rank of the pinched filling, since the slope is noncommensurable with the first manifold's longitude.

Question. Q4. What goes wrong if you slide a π-corner like a true corner? **Model answer.** The inside of the corner need not occupy less than half of the boundary tile. Then the antipodal argument no longer guarantees that every type (a) corner can be paired with a type (d) corner. A slide of a single straight corner may actually increase intersections. HRW instead reduces all true corners first and compares the remaining HF-solid-torus curve using an annular neighborhood (Figure 58).

Question. Q5. What is a Heegaard Floer homology solid torus? **Model answer.** Under the hypotheses of HRW22 Theorem 27 it is equivalently a bordered object invariant under a Dehn twist along the rational longitude, a knot exterior with all non-longitudinal fillings L-spaces, or an immersed-curve invariant with trivial local systems supported in a narrow neighborhood of the longitude. It need not be an ordinary solid torus; a twisted I-bundle over the Klein bottle gives a boundary-incompressible example.

Question. Q6. Why does an essential separating torus force rank at least five? **Model answer.** Its boundary-incompressible sides have nontrivial immersed-curve invariants. HRW's corner estimate gives at least four intersections unless the HF-solid-torus cases intervene. The exactly-four configurations are excluded by a fifth intersection or by an admissibility-induced extra Floer pair; HF-solid-torus cases are analyzed separately using tight and loose components. This is Theorem 1.8, not a consequence of the simple pinch inequality.

Question. Q7. Can you infer $HF(\Gamma_0, \Gamma_1) \cong HF(\Gamma_0, \hat{\Gamma}_1)$? **Model answer.** No. The loose-replacement construction changes puncture winding and may replace nontrivial local systems with trivial ones. Only minimal geometric intersection **inequality** is established. The pinching theorem uses this inequality and the ordinary gluing pairing, not an isomorphism between the two curve pairings.

Question. Q8. What does Theorem 1.12 say about all degree-one maps? **Model answer.** Nothing universal. It proves rank monotonicity for pinches on toroidal integer homology spheres. Theorem 7.15 proves a stronger *numerical* inequality for rational homology-sphere torus decompositions, and only in certain settings is there a degree-n map realizing the rational pinch. HRW's Question 1.13 asks about all degree-one maps; the pinching result does not answer it.

17. Reading assignment, proof checkpoints, and references

Four-hour-fifty-minute allocation

Time	Primary source	Required output
35 min	HRW23 Introduction, Theorems 1.8, 1.12; Section 7.2	State the two distinct rank theorems and longitude conventions
70 min	HRW23 Section 7.3, printed pp. 71–72, Figures 54–55	Reconstruct nm and the four-corner intersection change
50 min	HRW23 Section 7.3, printed pp. 73–74, Figures 56–58	Explain the cyclic charging proof and straight-corner exception
40 min	HRW23 Section 7.4, printed pp. 74–77	Trace the case division that raises rank four to five
35 min	HRW22 Section 1.5, Theorem 27; Section 5.2, Proposition 65	Explain HF-solid-torus curves and norm-detection restrictions
60 min	Exercises 1, 3, 5, 6 and oral questions	Produce written calculations and a theorem dependency map

Total: 4 hours 50 minutes.

Proof checkpoints

- ☐ Distinguish primitive rational longitude λ from homological longitude $n\lambda$.
- ☐ Derive $H_1(M, \partial M) \cong \operatorname{coker} j_*$.
- ☐ Explain exactly what n and m count, and why $n = m = 1$ in the integer case.
- ☐ Derive the two filling and gluing Floer intersection formulas.
- ☐ Prove the **algebraic** longitudinal multiplicity lower bound.
- ☐ List the five loose-replacement moves and explain why only corner slides are difficult.
- ☐ Produce the four local corner changes and $\Delta I = 2(N_a - N_d)$.
- ☐ Reconstruct the geometric antipodal/half-perimeter argument (Figures 56–57).
- ☐ Explain the π-corner case without incorrectly applying the true-corner argument.
- ☐ Write the four-step inequality yielding Theorem 7.15.
- ☐ Explain why exactly-four corner intersections do not contradict the rank-five theorem.
- ☐ Distinguish the main paper's Theorem 7.15 from the companion's HF-solid-torus characterization.
- ☐ State the arbitrary degree-one map question without claiming it is solved.

Verified references (paper/theorem/page audit)

[HRW23] J. Hanselman, J. Rasmussen, L. Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, arXiv:1604.03466v3 (2023), PDF. Theorem **1.12** and Question **1.13**, printed **p. 6**; Corollary **4.10**, Section 4.1; Corollary **6.6**, Section 6.2; Proposition **7.13**, printed **p. 70**; Section **7.3**, Theorem **7.15** and Figures **54–58**, printed **pp. 71–74**; Section **7.4**, proof of Theorem **1.8** and Figures **59–60**, printed **pp. 74–77**. HRW23's paper header dates its v3 text to March 8, 2023 and the arXiv version was posted April 25, 2023; these are not two different papers.

[HRW22] J. Hanselman, J. Rasmussen, L. Watson, *Heegaard Floer homology for manifolds with torus boundary: properties and examples*, *Proc. London Math. Soc.* **125** (2022), 879–967. Section **1.5**, Definition **26** and Theorem **27** on HF homology solid tori; Section **2** for gradings; Section **5.2**, Proposition **65** on the Thurston norm. The companion is the principal **second** HRW reference; the cylindrical/bordered reformulation is background only.

[LOT18, supporting] R. Lipshitz, P. Ozsvath, D. Thurston, *Bordered Heegaard Floer Homology*, *Memoirs AMS* **254** (2018), no. 1216, DOI 10.1090/memo/1216; use its pairing theorem and torus algebra as background, rather than as today's primary source.

Next lecture (Day 15, November 6): Reconstruct the rank-five theorem in finer detail using HRW23 Figures 59–60 and the companion's knot-Floer norm detection. Then analyze exceptional HF-solid-torus and Klein-bottle cases before moving to the satellite L-space knot application.