Day 15 - Sharp Toroidal Rank Bounds and Exceptional Curve Pairings

Friday, November 6, 2026. Study time: 4 h 50 min. **Field:** $\mathbb{F} = \mathbb{F}_2$. **Colocated PDF:** HRW_Day15.pdf. **Prerequisites:** HRW Day 06, HRW Day 13, HRW Day 14.

Concept links: Immersed Curves · Intersection Floer Homology · Pegboard Diagrams · Essential Tori · Spin-c Structures · Local Systems · Heegaard Floer Homology Solid Tori · Floer Admissibility · Thurston Norm.

> **Convention / caution** Verified source and scope **HRW23** means the three-author arXiv:1604.03466v3 (25 April 2023), whose **printed pp. 74–77** contain §7.4. The paper subsequently appeared in *Journal of the American Mathematical Society* **37 (2024), 391–498**, with different pagination. **HRW22** means the three-author published companion, *Proceedings of the London Mathematical Society* **125 (2022), 879–967**. Source theorems, elementary derived lemmas, and incomplete diagram-dependent sketches are explicitly distinguished. In particular, we will **not** claim that the crude four-intersection bound alone proves rank five.

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1. The exact theorem and why four is insufficient

Let Y be a closed connected oriented three-manifold containing a **separating essential torus** T. Cutting Y along T gives $Y = M_0 \cup_h M_1$, with boundary-incompressible torus boundaries. Denote by $\Gamma_i = \widehat{HF}(M_i)$ their **decorated curve invariants** in the punctured torus, carrying all local systems. We absorb the boundary identification and the standard elliptic correction into the placement of Γ_1.

> **Source theorem** Source: HRW23, Theorem 1.8 (introduction, p. 5; proof §7.4, pp. 74–77)
>
> $$\boxed{\dim_{\mathbb{F}} \widehat{HF}(Y) \geq 5.}$$
>
> The hypothesis is a **separating essential** torus, not just an arbitrary embedded torus. If Y is a rational homology sphere, every closed embedded orientable surface is separating (a nonseparating one would give a nonzero rational second-homology class); nonetheless we retain the source formulation.

By the geometric pairing theorem (HRW23, Theorem 1.2, with HRW22's elliptic-involution symmetry),

$$\widehat{HF}(Y) \cong HF(\Gamma_0, \Gamma_1).$$

Day 14 studied the preliminary corner estimate

$$i(\Gamma_0, \Gamma_1) \geq 2T + P + L,$$

where T, P, L count, respectively, **true** corners, **π-corners**, and loose components of Γ_1, *assuming Γ_0 is not of HF-homology-solid-torus type*. This is **equation (5)** in HRW23, p. 75. A nonstraight closed curve with any true corner has at least two true corners. Thus a first pass yields $i \geq 4$.

That estimate is not the theorem. The rest of today's lecture explains why every apparent rank-four configuration either has a fifth geometric intersection, violates manifold-specific homological restrictions, or requires an admissible Floer perturbation producing additional homology.

> **Convention / caution** Three quantities are not interchangeable $\#(\Gamma_0 \cap \Gamma_1)$ depends on a drawing; $i(\Gamma_0, \Gamma_1)$ is the minimum geometric number over allowable representatives; $\dim HF(\Gamma_0, \Gamma_1)$ is a homological invariant. Usually $\dim HF \geq i$, but in commensurable or annular configurations the inequality may be strict. Conversely, adding arbitrary intersection pairs does *not* automatically add Floer homology, since an acyclic differential may cancel them.

2. Why a separated torus forces two nontrivial bordered invariants

An essential separating torus is incompressible on either side. Each M_i is therefore boundary incompressible. HRW23, Proposition 7.13 proves the **manifold-specific** assertion

> **Source theorem** Source: HRW23, Proposition 7.13 (printed pp. 70–71) If the **entire** immersed-curve invariant $\widehat{HF}(M)$ is **loose**, then M is boundary compressible.

Thus neither side of an essential torus may have a wholly loose invariant. This does **not** rule out individual loose components; a combination of tight and loose components is exactly what occurs for the twisted I-bundle over the Klein bottle.

The geometric input behind Proposition 7.13 is knot Floer detection of a relative Thurston-norm quantity. For a rational longitude λ with order p in $H_1(M)$ and a dual meridian μ, a norm-minimizing S with boundary class $p\lambda$ is compared with the top nonzero Alexander level of the knot core of μ-filling. If the curve invariant were entirely loose, the observed height support would be too small unless the surface could be compressed. The full proof uses a doubled manifold and knot Floer norm detection: **incomplete proof sketch here**, see the original proof of Proposition 7.13.

There is a further homological restriction. If the primitive rational longitude λ has order p in $H_1(M)$, then the **homological longitude** is $p\lambda$. Corollary 6.6 of HRW23 constrains the total oriented class of the curves in each relevant Spin^c summand. The proof of Theorem 1.8 also invokes nonvanishing of the filling homology in at least p torsion Spin^c structures, furnishing additional components. These are restrictions on *manifold-derived* curves, not arbitrary curves in the Fukaya category. See §7.4, printed p. 74.

3. Tight and loose: source terminology and limitations

In the ϵ-pegboard construction (HRW23, §§7.1–7.2) the puncture is thickened to a forbidden disk. Geodesic portions of curves are tightened around its boundary. A **singular pegboard** arises as $\epsilon \to 0$. A corner is a **true corner** if the limiting curve changes tangent direction, and a **π-corner** if the limiting strand is straight across the peg.

A **loose** component is one that can be represented without essential pegboard corner behavior; a **tight** component cannot be simplified that way. We use these terms in their geometric sense, not to mean that one curve has trivial and the other nontrivial local-system monodromy. A **Heegaard Floer homology solid-torus curve** is one whose singular pegboard becomes a straight-line configuration. Such a collection can still include both tight and loose components in the punctured torus.

Two different statements are important:

- HRW23, Proposition 7.13: entirely loose $\Rightarrow$ boundary compressible (for a manifold invariant).
- HRW22, Theorem 27: under the theorem's **irreducibility and L-space knot-complement hypotheses**, HF-homology-solid-torus behavior is equivalent to twist invariance, all non-longitudinal fillings being L-spaces, and curves with trivial local systems living near the longitude.

It is **incorrect** to apply the second theorem to an arbitrary abstract extendable type D structure, or to assert that *every* component of a boundary-incompressible manifold must be tight.

4. The five-point lower bound begins with a corner ledger

> **Source theorem** Source inequality (HRW23, §7.4, equation (5), p. 75) If Γ_0 is **not** HF-homology-solid-torus type and Γ_1 has T true corners, P π-corners and L loose components (counted with the source's multiplicities), then
>
> $$\boxed{i(\Gamma_0, \Gamma_1) \geq 2T + P + L.}$$

Proof reconstruction (figure-dependent). Fix a true corner c_1 of Γ_1 and view a strand of Γ_0 in the square tile. The four endpoint configurations (a)–(d) were introduced in HRW23, Figure 55. A type-(d) corner alone gives two intersections. If type (a) occurs, the antipodal-endpoint argument of Day 14 produces a subsequent

type (d). A true type-(b) corner gives one intersection, and the strand must eventually produce another type (b) or (d) as it passes through possible straight type-(c) corners. A true type-(c) corner is a local maximum in a suitable direction; closure forces a local minimum, giving a type-(d) corner. Thus a fixed true c_1 gives **at least two intersections** with Γ_0.

For a π-corner the analogous analysis gives at least one intersection. Finally, a loose component avoiding Γ_0 would force Γ_0 into the same straight neighborhood, contrary to the hypothesis that it is not HF-solid-torus type. Summing the contributions gives the estimate. **Proof status:** the endpoint classification is checked in HRW23, p. 75; the derivation above omits the precise pictures and is a **geometric proof sketch**.

Why does a nonstraight closed curve have at least two true corners? In a singular pegboard, a genuine change of direction alters its directed segment slope. A closed polygonal itinerary cannot change direction exactly once and return to its original oriented tangent direction. HRW explicitly handle the apparent one-corner exception due to full winding around a puncture using Corollary 7.19. Thus the manifold setting permits us to use $T \geq 2$ whenever a true corner occurs.

5. The equality-four classification: two segments on each curve

Assume **neither** Γ_i is of HF-homology-solid-torus type. The corner estimate gives $i(\Gamma_0, \Gamma_1) \geq 4$. Suppose for contradiction that $i = 4$. Equality is so restrictive that each Γ_i must consist of precisely **one component with two true corners and two straight segments in the singular pegboard**. There can be no extra loose component or extra corner, and the local-system multiplicities are one. This is the rigidity step on printed p. 75.

Write the segment slopes as p_1, p_2 for Γ_0 and q_1, q_2 for Γ_1. Since neither curve is of HF-solid-torus type,

$$p_1 \neq p_2, \qquad q_1 \neq q_2.$$

The four intersections counted by the corner ledger are all near the puncture. Any further intersection **away** from the puncture violates equality. For any two rational slopes represented by primitive vectors $u = (a, b)$ and $v = (c, d)$, define their distance

$$\boxed{\Delta(u, v) = |ad - bc|.}$$

A straight segment of each slope has Δ intersections on the torus, **one** of which is already at the common puncture endpoint. Thus $\Delta - 1$ additional intersections persist away from the peg. Equality $i = 4$ forces

$$\boxed{\Delta(p_i, q_j) \leq 1 \quad (i, j \in \{1, 2\}).}$$

This is a strong arithmetic restriction, but it is not by itself a full classification: the two segments within each single closed immersed curve satisfy additional itinerary and homology constraints.

6. Slope distances and the three equality patterns

We need a simple arithmetic tool.

> **Proof / derivation** Derived determinant criterion For primitive integer vectors u, v, the oriented straight curves on the flat torus intersect minimally in $|\det(u, v)|$ points. Indeed, the subgroup $\mathbb{Z}u + \mathbb{Z}v \subset \mathbb{Z}^2$ has index $|\det(u, v)|$. Straight representatives have no bigons, so this realizes the minimum.

Normalize by an $SL(2, \mathbb{Z})$ change of coordinates. In the source's equality analysis there are three possible patterns, after also respecting the closed-curve itinerary.

Pattern A: four pairwise distinct segment slopes. HRW takes $p_1 = \infty$ and obtains, after the source's reparametrization, $p_2 = n$ and $q_1 = n - 1, q_2 = n + 1$. These slopes have pairwise cross-distances one to the opposite curve. The configuration forces the **rational longitude of one piece to have order two**. The homological and torsion-Spin^c restrictions then require an additional curve component, contradicting the assumed equality configuration. Thus Pattern A cannot produce rank four for a manifold gluing. (This is a source geometric classification step, not a consequence of determinant arithmetic alone.)

Pattern B: the two curves have the same two segment slopes. Up to reparametrization, $p = (\infty, 0)$ and $q = (\infty, 0)$. The curves coincide in the singular drawing. After a small perturbation there are **four** minimal geometric intersections. But the pairing contains an immersed annulus, so this four-generator picture is not an admissible Floer configuration. HRW find a necessary admissible perturbation producing **six-dimensional Floer homology**. This pattern therefore cannot give a rank-four toroidal gluing.

Pattern C: exactly one common segment slope. Up to reparametrization, $p = (\infty, 0)$ and $q = (\infty, 1)$. There is **one additional interior intersection** beyond the four corner intersections. Thus the pair has **five** intersections; Figure 59 realizes the sharp example.

Convention / caution An essential nuance of Pattern B Do **not** argue that adding two transverse intersections always increases Floer homology by two. It generally does not: acyclic bigon pairs may cancel. In Pattern B it is the *specific admissibility analysis of the immersed annulus* that gives the source's value $\dim HF = 6$. This is a nontrivial fact proved using the configuration, not rank-nullity alone.

7. The annular obstruction: four geometric intersections but six Floer generators

Why does commensurability matter? For noncommensurable curves in minimal geodesic position, the immersed-bigon differential vanishes and the rank is determined by geometric intersections, weighted by local-system ranks (HRW23, Corollary 4.10). For commensurable curves, periodic domains and annuli appear, as discussed in HRW23, §4.1.

Recall the admissibility condition (HRW23, Definition 4.7): when curves are commensurable, the relevant periodic domain must have both positive and negative coefficients. This ensures a finite positive-domain count. In a four-intersection minimal picture with coincident pegboard curves, the immersed annulus violates the needed condition. HRW's admissible modification adds two intersections, and in **that configuration** the Floer rank is six (p. 76).

The algebraic bookkeeping is worth separating from the geometry. If a finite complex has differential d, then

$$\boxed{\dim H(C,d) = \dim C - 2\,\mathrm{rank}(d).}$$

This follows from rank-nullity and $\operatorname{im} d \subset \ker d$. A hypothetical six-generator complex has Floer rank six **only if the differential vanishes**, whereas adding two generators to a four-generator complex together with a rank-one differential leaves homology dimension four. The special annulus argument is what decides the actual Floer result.

Source theorem Source: equality-case conclusion in HRW23, §7.4, printed p. 76 The identical two-segment configuration has $i(\Gamma_0,\Gamma_1) = 4$ but, after enforcing admissibility, $\dim HF(\Gamma_0,\Gamma_1) = 6$.

Proof status: The construction of the relevant annulus and the admissible modification is **diagram-dependent**. We retain it as a cited source result; the text above is a precise **incomplete proof sketch**, not a claim that an arbitrary coincident pair has this dimension.

8. Why the five-intersection pattern is sharp

In Figure 59 of HRW23 (printed p. 76), Γ_0 has two singular segments of slopes $0, \infty$, while Γ_1 has slopes $1, \infty$. The four corner contributions are supplemented by one genuine intersection away from the puncture. The geometric total is five.

The source realizes this curve pairing by gluing two **right-handed trefoil exteriors**, matching a 0-framed longitude on one side with a meridian and matching the other meridian with a 1-framed longitude (with the correct orientation reversal). The result is a **toroidal integer homology sphere** and

$$\boxed{\dim \widehat{HF}(Y) = 5.}$$

This proves **sharpness** of Theorem 1.8. It is different from the trefoil splice with $\dim \widehat{HF} = 7$ or the mixed-chirality splice with rank nine studied earlier in the course.

The signed intersection of the two oriented curve systems in Figure 59 is ± 1, compatible with the first-homology order one of the gluing. There are five Floer generators, and their mod-two Euler characteristic has absolute value one. As a derived consistency check, the only possible parity distributions of a five-dimensional $\mathbb{F}$-vector space with $|\chi| = 1$ are $(e,o) = (3,2)$ or $(2,3)$.

Do not identify total Floer rank with $|\chi|$: this example has rank five and Euler characteristic ± 1, hence is not an L-space.

9. Exactly one HF homology solid-torus piece

Assume M_1 has an HF-homology-solid-torus curve collection, while M_0 does not. HRW23, §7.4 first shows that a rank-four counterexample cannot involve true corners of Γ_1: a single component with exactly two true corners cannot meet the homological restrictions. If its segments have the same slope, it is either nullhomologous and requires another component, or represents twice a primitive class, and again requires additional Spin^c components.

We are left with π-corners and loose components. Let p be the order of the primitive rational longitude in $H_1(M_1)$. The homological and Spin^c restrictions force at least p relevant components, each contributing at least p intersection multiplicities with a nonstraight curve. In the case treated by HRW this yields

$$\boxed{i(\Gamma_0, \Gamma_1) \geq p^2.}$$

For $i < 5$, the source reduces to $p = 2$ and precisely two components, the special curves of the **twisted I-bundle over the Klein bottle**. One component must be tight, since the entire invariant cannot be loose for boundary-incompressible M_1 (Proposition 7.13). The loose component gives two intersections; careful inspection gives at least four more from the tight component, so

$$\boxed{i(\Gamma_0, \Gamma_1) \geq 6}$$

in this remaining exceptional configuration. This is the case analysis on HRW23, pp. 76–77; **the step ruling out the hypothetical other low-p arrangements uses the source's homological and pegboard arguments**.

10. Both sides HF homology solid tori

This is the subtlest portion of the theorem (HRW23, printed p. 77). Suppose both Γ_i pull tight to straight singular curves. They may contain tight and loose components. Since both M_i have incompressible boundary, Proposition 7.13 forces at least one **tight component** on each side.

The authors also prove that each side has a **loose component whose local system has eigenvalue** 1. This is not obvious from the pegboard picture. It follows by pairing with a closed curve L_i representing the rational longitude λ_i:

- Every tight component can be made disjoint from L_i without a relevant immersed annulus, so its Floer pairing with L_i is zero.
- For a primitive loose component with monodromy A, HRW23, Corollary 4.10 gives $\dim HF(\gamma_A, L_i) = 2 \dim \ker(I + A)$.
- The filled manifold $M_i(\lambda_i)$ has nonzero $\widehat{HF}$ (the paper uses the corresponding filling nonvanishing). Hence some loose component has $\ker(I + A) \neq 0$, equivalently 1 **is an eigenvalue of** A over $\mathbb{F}_2$.

This is the key place where local-system monodromy, even on a geometrically simple straight curve, is indispensable.

Let p_i be the order of the rational longitude in $H_1(M_i)$. For rational homology-sphere gluings with different longitude directions the source obtains strong product bounds, including $i(\Gamma_0, \Gamma_1) \geq p_0^2 p_1^2$ in the relevant setup. If both tight pieces consist only of π-corners, then $p_i \geq 2$, giving at least 16 intersections. The other exceptional small-p configurations have multiple traversals and likewise exceed four.

The remaining possibility of aligned longitudes illustrates why *Floer* rank cannot be replaced by geometric intersection number. A tight–tight pair supplies at least two intersections, and a loose–loose pair of monodromies with eigenvalue one contributes at least two **Floer homology** classes although the geometric minimal intersection can be zero. If the total lower bound were only four, there would be one tight and one loose component on each side and the tight components would share a commensurability class. Their Floer pairing then has dimension at least four after accounting for the relevant annulus. The overall rank is therefore at least six. This is depicted in **Figure 60**, using two twisted I-bundles over the Klein bottle.

Proof status: The full enumeration of exceptional components is **diagram-dependent** (HRW23, Figure 60 and §7.4, final paragraphs). The algebraic eigenvalue and kernel claims are proved next; the specific ≥ 6 tight-curve annulus estimate is a source result.

11. Local-system monodromy: explicit algebra and nonvanishing

Let $V = \mathbb{F}_2^r$ and $A \in GL(V)$ be the monodromy on an embedded primitive circle. Pair with a nearby parallel rank-one circle with trivial local system. The local Floer complex has two fiber copies, connected by the map $I + A$. Thus the dimension is

$$\boxed{\dim HF(\gamma_A, L) = 2 \dim \ker(I + A).}$$

This is the embedded, primitive, commensurable case of HRW23, Corollary 4.10. The number 2 arises because the two neighboring intersection families give a two-term complex with equal-dimensional source and target.

11.1. A rank-two unipotent example

Take

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It satisfies $\det J = 1$ and

$$I + J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence $\operatorname{rank}(I + J) = 1$, $\dim \ker(I + J) = 1$, and

$$\boxed{\dim HF(\gamma_J, L) = 2.}$$

This illustrates a loose component with eigenvalue one but **nontrivial monodromy**.

11.2. An irreducible rank-two counterpoint

Take

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $\chi_B(t) = t^2 + t + 1$, which has no root in $\mathbb{F}_2$. Moreover

$$I + B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \det(I + B) = 1.$$

Thus $I + B$ is invertible and

$$\boxed{HF(\gamma_B, L) = 0.}$$

So it would be false to assert that **every** loose component contributes nonzero pairing with the rational-longitude test curve. The filling nonvanishing argument isolates one with eigenvalue one.

11.3. Two loose components with eigenvalue one

Suppose $A_i \in GL(V_i)$ each has a nonzero eigenvector v_i of eigenvalue 1. Then

$$(A_0 \otimes A_1)(v_0 \otimes v_1) = v_0 \otimes v_1.$$

Thus

$$(I + A_0 \otimes A_1)(v_0 \otimes v_1) = 0.$$

The elementary tensor $v_0 \otimes v_1$ is nonzero, so

$$\boxed{\dim \ker(I + A_0 \otimes A_1) \geq 1.}$$

For the parallel primitive loose curves,

$$\boxed{\dim HF(\gamma_{A_0}, \gamma_{A_1}) \geq 2.}$$

This **complete algebraic proof** is the nonvanishing mechanism used in the final case of Theorem 1.8.

11.4. A self-pairing calculation

For self-pairing, the dual/Hom convention uses $A' = (A^{-1})^T$. Under $V \otimes V^* \cong \operatorname{End}(V)$, the kernel of $I + A \otimes (A^{-1})^T$ consists of endomorphisms commuting with A. Thus

$$\dim HF(\gamma_A, \gamma_A) = 2 \dim \operatorname{Cent}(A)$$

for embedded primitive curves. This verifies in another way that local systems can contribute Floer classes even when the minimal number of geometric intersections is zero. It is **not** an assertion that all two-component pairings are self-pairings.

12. How HRW22 complements the rank-five proof

The companion HRW22, §1.5, gives an intrinsic characterization of **Heegaard Floer homology solid tori**. Its Definition 26 requires the bordered differential modules to be invariant under changing a chosen slope dual to the rational longitude by a longitude Dehn twist. Its Theorem 27, **under the hypothesis that M is an irreducible knot complement in an L-space**, makes this equivalent to (a) every non-longitudinal filling being an L-space, and (b) the immersed curves being regular-homotopic into a small longitude neighborhood with **trivial local systems**. See also **Figure 16** for the twisted I-bundle over the Klein bottle.

At first this seems in tension with HRW23, §7.4, which allows an eigenvalue-one **nontrivial** local system on a loose component while analyzing abstract HF-solid-torus-like singular configurations. There is no contradiction: the full geometric classification on the manifold subclass in Theorem 27 has extra hypotheses, and HRW23 deliberately allows broader components while proving a universal lower bound. We must keep 'a *curve* with straight singular pegboard' separate from 'the *entire manifold invariant* satisfies Theorem 27'.

HRW22, Corollary 28, proves that when a closed manifold has an HF homology solid torus in a JSJ decomposition, repeated longitude twists can yield an infinite family of (possibly distinct) manifolds with isomorphic ungraded $\widehat{HF}$. This is a useful conceptual contrast: an exceptionally simple bordered object can make the *gluing homology* insensitive to some mapping-class changes while still contributing nontrivially to the essential-torus rank estimate.

The companion's §5.2 (Proposition 65) interprets the Thurston norm using the highest obstructed peg, and supplies intuition for why a curve collection cannot be wholly loose when the manifold has incompressible boundary. The actual non-looseness theorem used today is **HRW23, Proposition 7.13**, proved using knot Floer norm detection. LOT's bordered formalism is only a supporting input via the geometric pairing theorem, not today's focus.

13. Worked computations and genus/topology checks

13.1. A determinant table for the sharp pattern

Write $\infty = (1,0)$, $0 = (0,1)$, $1 = (1,1)$ as primitive vectors. Let $p = (\infty, 0)$ and $q = (\infty, 1)$. The cross-distances are

Δ	$q_1 = \infty$	$q_2 = 1$
$p_1 = \infty$	0	1
$p_2 = 0$	1	1

.

They are all at most one. Thus the off-peg intersection count *could* vanish except for the nonparallel straight segments and their endpoint identifications; HRW's Figure 59 shows the actual fifth transverse intersection. The determinant table is a **necessary check**, not a substitute for the diagram.

13.2. An elementary annulus and the Euler characteristic

If $\dim HF(Y) = 5$ and Y is an integer homology sphere, then $|\chi(\widehat{HF}(Y))| = |H_1(Y)| = 1$. Let e, o be the even and odd dimensions. We have

$$e + o = 5, \qquad |e - o| = 1,$$

so $(e, o) = (3, 2)$ or $(2, 3)$. Thus a sharp rank-five example necessarily has **both parities** and is not an L-space. In contrast, in the hypothetical rank-four chain picture with $\chi = 1$, no nonnegative integer parity decomposition is possible at the *homology* level: $e + o = 4$ and $e - o = \pm 1$ are inconsistent. This parity argument **alone** rules out rank four for an integer homology sphere, but **not** for a general rational homology sphere whose $|H_1|$ is even. The full pegboard theorem handles both.

13.3. Kernel ranks for two monodromies

Let $J = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We have $J^2 = I$. In a product basis,

$$J \otimes J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$I_4 + J \otimes J = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Over $\mathbb{F}_2$ its rank is two, so its kernel has dimension two. For **parallel primitive** loose components with monodromy J on both sides, the local-system contribution has dimension

$$\boxed{2 \dim \ker(I_4 + J \otimes J) = 4.}$$

This is a fully worked rank calculation; it does **not** replace the tight–tight contributions in Figure 60.

13.4. The source's two contrasting examples

HRW23, Figure 59 realizes $\dim \widehat{HF} = 5$ with a gluing of trefoil exteriors: the theorem's minimum is sharp. Figure 60 pairs two twisted I-bundles over the Klein bottle. There are at first two loose–loose and two tight–tight intersections; imposing the relevant admissibility creates additional intersections, and the pairing rank is six. The second picture is a geometric illustration of why an apparent four-intersection lower bound cannot be accepted as a completed Floer computation.

14. What the proof does not imply

Convention / caution Frequent oral-exam errors **(1)** The estimate $i \geq 2T + P + L$ assumes one side is not HF-solid-torus type; it does not apply verbatim in the both-exceptional case. **(2)** A single loose curve does not imply boundary compressibility: Proposition 7.13 concerns the *whole* invariant. **(3)** A local system with monodromy A contributes $2 \dim \ker(I + A)$ to the rational-longitude test; this can be zero. **(4)** Coincident straight segments need an admissibility analysis; a formal count of four drawn crossings is insufficient. **(5)** The rank-five bound applies to **separating** essential tori in arbitrary closed oriented Y; the stronger classification of small **toroidal L-spaces** uses additional hypotheses and HRW23, Theorem 7.20. **(6)** Theorem 27 of HRW22 has manifold hypotheses; do not apply it to arbitrary curve objects. **(7)** Equality $\dim HF = 5$ does not imply the glued manifold is an L-space, as the sharp integer-homology-sphere example shows.

A useful derived consequence of Theorem 1.8 is: if Y is a prime rational homology sphere with $\dim \widehat{HF}(Y) < 5$, then Y is atoroidal; geometrization then restricts the possible geometric structures (HRW23, Corollary 1.9). If Y is an integer homology-sphere L-space, then its Floer rank is one, hence no separating essential torus can occur (HRW23, Corollary 1.10). These are genuine consequences, not part of the equality-four diagram analysis.

15. Exercises (eight serious problems)

Exercise 1 — The coarse bound and its exact hypotheses State HRW23's equation (5) with the hypothesis on Γ_0 and the meaning of T, P, L. Explain how two true corners give $i \geq 4$. Why does $i \geq 4$ fail to establish Theorem 1.8? Reproduce, using Figure 55, the four local corner-type contributions and explain the special role of type (d). **Hint:** Analyze one fixed true corner of Γ_1 and follow the corners of the closed curve Γ_0 cyclically.

Exercise 2 — Rigidity of equality Assume both curve collections are not HF-solid-torus type and $i = 4$. Reconstruct why equality forces one component with exactly two true corners on each side, with trivial rank-one local systems. Let their segment slopes be p_1, p_2 and q_1, q_2. Explain why $\Delta(p_i, q_j) \leq 1$ is necessary. Determine the cross-distance matrix for $p = (\infty, 0)$, $q = (\infty, 1)$. **Hint:** An intersection of straight segments away from the puncture would be an uncounted fifth point.

Exercise 3 — Annulus versus a chain-level cancellation Suppose an immersed-curve drawing has four minimal intersections but contains a periodic annulus. Explain why this may fail HRW's admissibility condition. State the source's actual result for the coincident two-segment configuration. Separately, calculate homology dimensions for a complex of dimension six with differential of ranks $0, 1, 2, 3$. Why is the phrase 'adding two crossings raises Floer rank by two' unjustified in general? **Hint:** Use $\dim H = \dim C - 2 \operatorname{rank} d$.

Exercise 4 — Nontrivial local systems on loose circles Work over $\mathbb{F}_2$. For $J = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, calculate $\chi_J(t), \chi_B(t)$, $\dim \ker(I + J)$, and $\dim \ker(I + B)$. Find the dimension of the Floer pairing with a trivial rank-one parallel longitude test circle. Explain why the presence of a loose component **alone** is insufficient for the rational-longitude filling nonvanishing argument. **Hint:** Eigenvalue one is equivalent to $\ker(I + A) \neq 0$.

Exercise 5 — Product monodromy Assume $A_0 \in GL(V_0)$ and $A_1 \in GL(V_1)$ each have eigenvalue one. Prove directly that $I + A_0 \otimes A_1$ is singular over $\mathbb{F}_2$. Then compute $I_4 + J \otimes J$ for the unipotent J from Exercise 4; determine its rank and nullity. Give the associated primitive-loose-circle Floer dimension. **Hint:** Tensor eigenvectors and then compute with a fixed tensor-product basis.

Exercise 6 — The one-exceptional-side case Read HRW23, pp. 76–77. Suppose M_1 is HF-solid-torus type and M_0 is not. Explain why the equality case with a single curve of two true corners is excluded, and why the remaining straight-π-corner case is constrained by the order p of the rational longitude. Analyze the $p = 2$ exceptional pattern and explain how a loose component plus a tight component forces at least six intersections. **Hint:** Use Proposition 7.13 and Figure 53, then check the slope of the two segments of Γ_0.

Exercise 7 — Both sides exceptional and HRW22 Reconstruct the two separate nonvanishing arguments when both M_i have HF-solid-torus-type curves: why each must have a tight component,

and why each must have a loose component whose monodromy has eigenvalue one. State the precise manifold hypotheses in **HRW22, Theorem 27**. Compare the general HRW23 Figure 60 local-system analysis with HRW22's global Theorem 27; is there a contradiction? **Hint:** Theorem 27 concerns the *entire invariant* of an irreducible knot complement in an L-space.

Exercise 8 — Sharpness and consequences Use Figure 59 to identify two segment-slope sets giving the sharp rank-five example. Explain why its first homology is trivial but its Floer rank is five. Deduce the mod-two parity dimensions of its Floer homology. Prove HRW23, Corollaries 1.9 and 1.10 from Theorem 1.8 together with the stated topological hypotheses. Finally explain why the pinching theorem of Day 14 cannot itself imply the rank-five theorem. **Hint:** For a rational homology sphere $|\chi\widehat{HF}| = |H_1|$; compare rank and Euler characteristic.

16. Hints and selected solutions

16.1. Exercise 1

Equation (5) is

$$i(\Gamma_0, \Gamma_1) \geq 2T + P + L$$

when Γ_0 is **not** HF-solid-torus type. If Γ_1 has a true corner it has at least two, so $i \geq 4$. This is insufficient because $4 < 5$. The key local mechanism is that a type-(d) corner of Γ_0 supplies two intersections with a fixed true c_1, and the other configurations force a type-(d) or another positive contribution upon following the closed curve. The missing diagrams are Figures 55–57 (proof of §7.4 on p. 75).

16.2. Exercise 2

For $\infty = (1,0)$, $0 = (0,1)$, $1 = (1,1)$,

$$\Delta(\infty,\infty) = 0, \quad \Delta(\infty,1) = 1, \quad \Delta(0,\infty) = 1, \quad \Delta(0,1) = 1.$$

Thus the cross-distance matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The numerical test says no extra intersection is *forced* by a distance ≥ 2; the fifth point follows from the actual arrangement and the shared end/corner structure in Figure 59, not from this table alone.

16.3. Exercise 3

If $\dim C = 6$, the possible homology dimensions with $d^2 = 0$ and $\operatorname{rank} d = r$ are

$$6 - 2r \in \{6, 4, 2, 0\} \quad (r = 0, 1, 2, 3).$$

An admissible perturbation introducing two chain generators can keep homology unchanged if it also introduces a rank-one acyclic differential. The **specific** HRW coincident-curve configuration has $i = 4$ but $\dim HF = 6$; that conclusion is supplied by the immersed-annulus analysis, not by the rank-nullity identity.

16.4. Exercise 4

$\chi_J(t) = (t+1)^2$ and $\chi_B(t) = t^2 + t + 1$. We have

$$I + J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad I + B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The nullities are one and zero, respectively. Hence the two parallel trivial-test pairings have dimensions two and zero. A rank-two loose component need not contribute to the rational-longitude filling: its monodromy must have eigenvalue one.

16.5. Exercise 5

For eigenvectors $A_i v_i = v_i$ with $v_i \neq 0$, $v_0 \otimes v_1$ is a nonzero 1-eigenvector for $A_0 \otimes A_1$; it lies in $\ker(I + A_0 \otimes A_1)$. For J, the matrix in §13.3 has two independent nonzero rows, so rank two and nullity two. The parallel primitive loose pairing has dimension $2 \cdot 2 = 4$.

16.6. Exercise 6 — Solution outline

The all-straight case of the second collection has p relevant components, each contributing at least p intersections, so a putative count below five forces the source's $p = 2$ exceptional situation. The complement is boundary incompressible and its whole invariant cannot be loose, so at least one component is tight. Figure 53 identifies the relevant twisted-I-bundle pattern. If both segment slopes of Γ_0 differ from the rational longitude, the tight component supplies two type-(d) corners; if one agrees, there are two extra off-peg intersections. Added to the two intersections with the loose component, the result is at least six. **The local figure verification is required**.

16.7. Exercise 7 — Solution outline

Tight components are forced by Proposition 7.13. The rational-longitude filling pairing is nonzero; tight components have zero pairing with that test, and loose components with monodromy A contribute $2\dim\ker(I + A)$. Thus some loose A has eigenvalue one. HRW22, Theorem 27 assumes an oriented compact connected **irreducible** knot complement in an L-space. The theorem's trivial-local-system conclusion is about this whole manifold subclass; HRW23's componentwise exceptional algebra is broader. There is no contradiction.

16.8. Exercise 8

For Figure 59 the slopes are $(\infty, 0)$ and $(\infty, 1)$. The source states the glued manifold is an integer homology sphere (signed geometric intersection ± 1), and its Floer rank is five. Since the Euler characteristic has absolute value one, $e + o = 5$ and $|e - o| = 1$; therefore $(e, o) = (3, 2)$ or $(2, 3)$. An integer homology-sphere L-space has rank one and cannot contain a separating essential torus (Corollary 1.10). For a prime rational homology sphere with rank below five, every essential torus would separate, so there is none; the manifold is atoroidal, and geometrization gives the conclusion of Corollary 1.9. The pinch theorem only gives comparison with a **particular filling**, which might have rank one; it cannot force rank five.

17. Oral examination with model answers

Oral 1 — Which step is missing between four and five? Model answer: Equation (5) gives at least four intersections if both collections are non-HF-solid-torus type. The equality-four case forces one two-segment curve on each side and rank-one local systems. The determinant constraints leave three geometric patterns: four distinct slopes forces an order-two longitude and extra components; identical slopes creates an inadmissible annulus and Floer rank six; one common slope gives the sharp five-intersection Figure 59. The special HF-solid-torus cases must be treated separately.

Oral 2 — What does $\Delta(p, q) \leq 1$ mean here? Model answer: It says no additional intersection point is forced in the interiors of two singular straight segments, beyond the meeting near their puncture endpoint. All four corner contributions have already been counted. The determinant condition is necessary for equality four, but is not sufficient: a pair of segments must also assemble into permitted closed immersed curves, satisfying homology and local-system constraints.

Oral 3 — Why is an immersed annulus a problem? Model answer: Commensurable curves admit periodic annular domains. A minimal drawing can fail the admissibility condition needed for finite, invariant polygon counting. The coincident two-segment equality picture has four minimal intersections but an annulus; HRW's admissible modification produces Floer dimension six. One cannot infer Floer dimension directly from the original four crossings.

Oral 4 — Why is eigenvalue one forced on a loose component? Model answer: Pair the invariant with the rational-longitude test circle. Its Floer homology computes a nonzero filling group. Tight components pair trivially, while a primitive loose component with monodromy A contributes $2\dim\ker(I + A)$. Therefore a loose component must have nonzero kernel, equivalently eigenvalue one over $\mathbb{F}_2$.

Oral 5 — Why is Proposition 7.13 not 'there are no loose components'? Model answer: It says an **entirely loose** manifold-derived invariant forces boundary compressibility. An incompressible-boundary manifold may have some loose components, and in the both-HF-solid-torus case HRW require both tight and loose components. Confusing a componentwise statement with a global statement would invalidate the proof.

Oral 6 — What does the twisted I-bundle example illustrate? Model answer: It is boundary incompressible but has HF-homology-solid-torus behavior. Its invariant contains exceptional tight/loose geometry. Figure 60 shows that an apparent four intersections becomes Floer rank six after admissibility, illustrating why the both-exceptional case needs separate analysis.

Oral 7 — Is the sharp rank-five example an L-space? Model answer: No. HRW's Figure 59 gives a toroidal integer homology sphere with Floer rank five. An integer-homology-sphere L-space would have rank one. Its even/odd Floer dimensions are three and two in some order, giving Euler characteristic ± 1.

Oral 8 — How do the two HRW papers divide the work? Model answer: HRW23, §7.4 supplies the full pegboard and local-system proof of the sharp rank-five theorem, based on its classification and pairing theory. HRW22, §1.5, precisely characterizes HF homology solid tori under manifold hypotheses and gives Figure 16 and the twist-invariance viewpoint. The companion's norm detection explains why the full curve set cannot be loose; the source of that exact non-looseness theorem is HRW23, Proposition 7.13.

18. Reading and proof checkpoints

Total: 4 hours 50 minutes. The paper has been read once already, so prioritize reconstructing proof *mechanisms* rather than passively rereading all of §7.

Time	Assignment	Required result
35 min	HRW23 §7.4, printed p. 74; Proposition 7.13	State non-looseness and identify manifold-only restrictions
55 min	HRW23 §7.4, printed p. 75; Figure 55	Reconstruct the $2T + P + L$ estimate
65 min	HRW23 printed p. 76; Figure 59	Explain every equality-four pattern and the annular exceptional case
50 min	HRW23 printed pp. 76–77; Figure 60	Reconstruct one-/two-HF-solid-torus cases
35 min	HRW22 §1.5, Definition 26, Theorem 27, Figure 16	Compare global HF solid tori with componentwise loose/tight pieces
50 min	Exercises 3–5, 7–8; oral practice	Compute kernels and state complete proof map

Proof checkpoints:

- ☐ Derive the two-intersection contribution of a true corner using the four endpoint types.
- ☐ Explain why equation (5) applies only if one collection is not HF-solid-torus type.
- ☐ Derive the equality-four rigidity: one two-segment component on each side.
- ☐ Verify $\Delta(p_i, q_j) \leq 1$ and explain why it is necessary but not sufficient.
- ☐ Explain all three equality patterns and the role of the order-two rational longitude.
- ☐ Distinguish $i = 4$ from $\dim HF = 6$ in the annular pattern.
- ☐ Reproduce the rank-five Figure 59 slope sets and check parity/Euler characteristic.
- ☐ Explain why the one-exceptional-side case contributes at least six.
- ☐ Prove the $2 \dim \ker(I + A)$ formula in the local rank-two examples.
- ☐ Prove eigenvalue-one nonvanishing under tensor products.
- ☐ Reconstruct the both-exceptional tight/loose and admissibility argument.
- ☐ State HRW22, Theorem 27 with its correct hypotheses.
- ☐ Explain why the pinch inequality does not imply the rank-five theorem.

19. Sources and status of proofs

[HRW23, main] J. Hanselman, J. Rasmussen, L. Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, arXiv:1604.03466v3 (dated 25 April 2023); final publication *Journal of the American Mathematical Society* **37 (2024)**, 391–498. **Use the arXiv v3's printed pagination:** Theorem 1.8 (p. 5); Proposition 7.13 (§7.2, printed pp. 70–71); §7.4 (printed pp. 74–77), equation (5), Figures 59–60; Theorem 7.15 (§7.3); Corollaries 4.10, 6.6, and 7.19. The inequality $i \geq 2T + P + L$, equality-case classification and the exact six-dimensional annular computations are **source geometric arguments**, only partly reconstructed above.

[HRW22, companion] J. Hanselman, J. Rasmussen, L. Watson, *Heegaard Floer homology for manifolds with torus boundary: properties and examples*, *Proceedings of the London Mathematical Society* **125 (2022)**, 879–967. Main assignments: §1.5, Definition 26, Theorem 27, Corollary 28, Figure 16; §5.2, Proposition 65 (supporting relation to the Thurston norm). **Numbering here uses the published edition.**

[**LOT18, supporting**] R. Lipshitz, P. Ozsváth, D. Thurston, *Bordered Heegaard Floer Homology*, *Memoirs AMS* **254 (2018)**, no. 1216. Bordered pairing is used through HRW's immersed-curve theorem, not developed anew here.

[**Proof-status audit**] Complete within this lecture: determinant intersection arithmetic, rank-nullity, parity constraints, kernel and Kronecker-product calculations, the eigenvalue-one tensor lemma, conditional consequences of rank five. **Incomplete geometric proof sketches**, requiring the cited figures: the $2T + P + L$ endpoint-by-endpoint corner classification, the equality-four itinerary enumeration, the immersed annulus admissibility perturbation in Figure 59, and the tight/loose exceptional pairings in Figure 60. No independent claim of having reproved these source results without their diagrams is made.

Next day (Day 16, 7 November): HRW23, §§7.5–7.7 revisited as an oral-exam synthesis, then satellite L-space knots and tangent-slope obstructions, with HRW22 filling-core Floer theory as the companion reference.