Day 16 - Satellite L-Space Knots: Pattern, Companion, and Gluing

Saturday, November 7, 2026 | Rigorous English HRW qualifying-exam course | 4 h 50 min
Prerequisites: L-Space Gluing Theorem, Dehn Filling, Immersed-Curve Pairing, Knot Floer Homology, Turaev Torsion.
PDF edition: This lecture is the printable version of the companion Obsidian note.
Primary texts: HRW23, arXiv v3, §7.8, printed p. 84, together with Theorems 1.14 and 7.17; HRW22, published companion, §4 and §5.3. The printed page 84 is page 84 of the *arXiv PDF* (zero-based page 83), not a page of the later JAMS pagination.

Reading map and contents

- 1. The geometric setting: which torus, which exterior, which slope?
- 2. Verified source theorems and logical dependencies: source results versus deductions.
- 3. Two slopes and two fillings: the pattern-complement diagram.
- 4. A homology lemma: longitude filling is not an L space: complete Mayer-Vietoris calculation.
- 5. The satellite theorem, step by step: substantial proof and one explicitly marked external step.
- 6. The 2022 companion: why immersed curves enter: knot Floer and grading inputs.
- 7. Worked calculation I: a positive trefoil cable: Alexander polynomial and L-space interval.
- 8. Worked calculation II: necessity is not sufficiency: a cable below the threshold.
- 9. Worked calculation III: a non L space companion: figure-eight obstruction.
- 10. Convention and logical audit: hypotheses that cannot be suppressed.
- 11. Proof checkpoints; 12. Exercises with hints; 13. Solutions; 14. Oral examination; 15. Four hour fifty minute plan; 16. Verified references.

1. The geometric setting

Let $V = D^2 \times S^1$ be an **unknotted** solid torus in S^3. A pattern is a smooth embedded oriented knot $P \subset \mathrm{int}(V)$. Let $C \subset S^3$ be a nontrivial companion knot with a specified zero-framed tubular neighborhood νC. An orientation-preserving embedding $\varphi : V \to \nu C$ identifying preferred longitudinal directions produces the satellite knot $K = P(C) = \varphi(P)$. We assume the companion torus $T = \partial \nu C$ is **essential** in the exterior of K; this is the usual nondegenerate satellite situation, and is needed when applying the boundary-incompressible gluing theorem. The full implication of HRW23, Theorem 1.15 concerns satellite knots; the proof below chooses a surgery retaining the essential torus.

Use three exteriors, which must never be confused:

$$X_C = S^3 \setminus \mathrm{int}(\nu C), \qquad W = V \setminus \mathrm{int}(\nu P), \qquad X_K = S^3 \setminus \mathrm{int}(\nu K).$$

The pattern exterior W has **two** torus boundary components, $T_{\mathrm{in}} = \partial V$ and $T_{\mathrm{out}} = \partial \nu P$. The satellite knot exterior has the decomposition

$$\boxed{X_K = X_C \cup_h W,}$$

with $h : T_{\mathrm{in}} \to \partial X_C$ induced by φ. On the companion side write μ_C for the meridian and λ_C for the zero-framed Seifert longitude. Write $\ell_V = \{\mathrm{pt}\} \times S^1 \subset \partial V$ for the **core-longitude of the solid torus**. Its image under the zero-framed satellite identification is λ_C (up to orientation). Set $\alpha_P = \ell_V \subset T_{\mathrm{in}}$. Hence

$$\boxed{h(\alpha_P) = \lambda_C.}$$

The pattern knot **in** S^3, denoted $P(U)$ (often abbreviated P), is obtained by placing V as an unknotted solid torus. Note that $P(U)$ need not be the core of V and may even be the unknot. The winding number $w \in \mathbb{Z}$ is defined by $[P] = w[\ell_V]$ in $H_1(V; \mathbb{Z})$; our theorem does not assume a particular value of w.

> **WARNING** Satellite notation An L-space assertion about the **pattern knot** means $P(U) \subset S^3$, not the pattern regarded merely as a curve in V. Also, W has two boundary components and is *not* eligible directly as either torus-boundary input of HRW23, Theorem 1.14. Fill its **outer** boundary first.

2. Verified source theorems and logical dependencies

> **THEOREM** HRW23, Theorem 1.15 (source theorem; §7.8, p. 84) If a satellite knot $K = P(C) \subset S^3$ is an L-space knot, then both C and the pattern knot $P(U)$ are L-space knots. Here an L-space knot admits a **nonmeridional, nontrivial** Dehn surgery yielding an L-space.

HRW23 explicitly identifies this as the resolution of Hom–Lidman–Vafaee, Conjecture 1.7 (appearing in their paper as Conjecture 7.21). **This is necessary, not sufficient, for $P(C)$ to be an L-space knot.** Source: HRW23, introduction Theorem 1.15; §7.8.

> **THEOREM** HRW23, Theorem 1.14 (source theorem; §7.6) If $Y = M_0 \cup_h M_1$ is obtained from **boundary-incompressible** compact oriented manifolds with one torus boundary each using an orientation-reversing homeomorphism, then
>
> $$\boxed{Y \text{ is an L-space} \iff \mathcal{L}^\circ(M_0) \cup h(\mathcal{L}^\circ(M_1)) = \mathbb{Q}P^1.}$$
>
> Here $\mathcal{L}^\circ$ denotes the **interior** of the rational L-space filling slopes in the projective slope circle.
>
> **THEOREM** HRW23, Theorem 7.17 (source theorem; §7.5) The interior L-space slopes are the rational slope directions that avoid the essential singular tangent-slope obstruction of the tightened immersed-curve invariant. This is the geometric input behind (2.1), not an assertion about arbitrary curves without manifold provenance.
>
> **THEOREM** HRW22, Theorem 11 and Proposition 12 (source results; §4) For a knot $C \subset S^3$, its $CFK^-(C)$ determines the decorated immersed-curve invariant $\widehat{HF}(X_C)$. Conversely, pairing a torus-boundary curve invariant with the proper noncompact test arc of a filling slope recovers $\widehat{HFK}$ of the filling core, with a relative Spin^c refinement (Proposition 12).

Dependency, not equivalence of statements: HRW22 reconstructs knot Floer information and describes the graded geometry. HRW23 supplies the gluing criterion. The satellite proof in §7.8 is extremely short precisely because it uses the already established gluing theorem; it does *not* compute a new satellite knot Floer complex.

3. Two slopes and two fillings

Perform a p/q-surgery along K (coprime p, q with $p \neq 0$), and denote the result by $Y_r = S^3_r(K)$ where $r = p/q$. It is crucial to fill T_{out}, not T_{in}. Define

$$\boxed{M_r = W \cup_{T_{\text{out}}} (D^2 \times S^1)_r.}$$

Thus M_r is a compact oriented three-manifold with the single remaining boundary T_{in}, and

$$\boxed{Y_r = X_C \cup_h M_r.}$$

Because the two filling tori are disjoint, filling them in either order gives the same closed manifold:

$$\begin{aligned} M_r(\alpha_P) &= W \cup_{T_{\text{out}}} (D^2 \times S^1)_r \cup_{T_{\text{in}}} (D^2 \times S^1)_{\alpha_P} \\ &\cong (W \cup_{T_{\text{in}}} (D^2 \times S^1)_{\alpha_P})(r) \\ &\cong \boxed{S^3_r(P(U)).} \end{aligned}$$

The last homeomorphism uses that $\alpha_P = \ell_V$ is the meridian of the **complementary solid torus** $S^3 \setminus \text{int}(V)$. Filling T_{in} along ℓ_V converts W into the exterior of the pattern knot in S^3. Equation (3.3) is a homeomorphism of *ungraded topological surgery manifolds*, not a claim of a canonical identification of every Floer grading.

> **WARNING** The crucial longitude/meridian switch The slope α_P is the **longitude** of V, but it is the **meridian of the solid torus complementary to** V in the standard genus-one splitting of S^3. Filling the inner boundary of W along μ_V instead does **not** recover the pattern exterior in S^3.

4. A homology lemma: longitude filling is not an L-space

THEOREM Derived Lemma 4.1 (complete proof) If $C \subset S^3$ is any knot, then $S^3_0(C)$ has $H_1 \cong \mathbb{Z}$. Thus the Seifert longitude λ_C is not an L-space filling slope of X_C.

PROOF Alexander duality or the standard meridian calculation gives $H_1(X_C) \cong \mathbb{Z}\langle\mu_C\rangle$. The preferred longitude is null-homologous in X_C: $[\lambda_C] = 0$. The effect of p/q-filling on first homology is to impose the relation $p[\mu_C] + q[\lambda_C] = p[\mu_C] = 0$. Therefore

$$H_1(S^3_{p/q}(C)) \cong \begin{cases} \mathbb{Z}/|p|, & p \neq 0, \\ \mathbb{Z}, & p = 0. \end{cases}$$

In particular, longitude filling is $p = 0$, and $b_1(S^3_0(C)) = 1$. By definition an L-space is a **rational homology sphere** with minimal hat Floer rank, so $S^3_0(C)$ is not an L-space. □

We emphasize that this does not require C to be fibered, positive, an L-space knot, or even nontrivial. The nontriviality hypothesis enters through **boundary incompressibility** when applying the gluing theorem.

5. The satellite theorem, step by step

The following is a careful reconstruction of the proof of HRW23, Theorem 1.15 (arXiv v3, printed p. 84), with the topological surgery-selection input separated.

5.0. Why M_r is a rational homology solid torus

PROOF Derived Mayer–Vietoris rank lemma (complete proof) The knot surgery $Y_r = S^3_r(K)$ with nonzero numerator is a rational homology sphere by Lemma 4.1. In the splitting $Y_r = X_C \cup_h M_r$, both sides are connected. The rational Mayer–Vietoris sequence contains

$$H_1(T;\mathbb{Q}) \longrightarrow H_1(X_C;\mathbb{Q}) \oplus H_1(M_r;\mathbb{Q}) \longrightarrow H_1(Y_r;\mathbb{Q}) = 0.$$

Therefore the first arrow is surjective. The source has dimension 2, and $H_1(X_C;\mathbb{Q})$ has dimension 1, so $b_1(M_r) \leq 1$. A connected compact oriented three-manifold with exactly one torus boundary component has $\chi = 0$ (double it), and hence $1 - b_1 + b_2 = 0$ over $\mathbb{Q}$. Thus $b_1(M_r) = 1 + b_2(M_r) \geq 1$. We obtain $b_1(M_r) = 1$ and $b_2(M_r) = 0$: M_r is a rational homology solid torus. Its rational-longitude filling therefore has $b_1 = 1$ and is not an L-space. □

This rank lemma is needed when asserting that neither interior L-space interval in the gluing can equal the entire projective slope circle.

5.1. Selecting an L-space surgery that retains the companion torus

Since K is an L-space knot, it admits an interval of nontrivial L-space slopes; after choosing chirality, it contains arbitrarily large positive integer slopes. To invoke (2.1), choose such a slope r for which the image of the essential companion torus remains incompressible in Y_r; equivalently, M_r has incompressible boundary. The standard **persistence of essential tori under all sufficiently generic Dehn fillings** provides this choice in the satellite setting. (A cable pattern can have an exceptional cabling slope; one does not choose it.)

PROOF External topological input – proof sketch, not proved here The torus T is incompressible on the companion side because C is nontrivial and on the pattern side by the essential satellite hypothesis. Incompressibility can be destroyed at exceptional outer-boundary filling slopes, but not throughout the infinite tail of L-space slopes. The source proof begins by choosing a **toroidal L-space surgery**, rather than proving this persistence lemma in §7.8. We use that choice as a standard topological input, and do not present a general Dehn-filling persistence theorem without its extra hypotheses.

With this choice, both X_C and M_r satisfy the boundary-incompressibility hypotheses of Theorem 1.14.

5.2. First conclusion: the companion is an L-space knot

PROOF Complete deduction from Theorem 1.14 By construction, $Y_r = X_C \cup_h M_r$ is an L-space and both pieces have incompressible torus boundary. Theorem 1.14 implies

$$\boxed{\mathcal{L}^\circ(X_C) \cup h(\mathcal{L}^\circ(M_r)) = \mathbb{Q}P^1.}$$

If $\mathcal{L}^\circ(X_C)$ were empty, the right-hand set would have to be contained in $h(\mathcal{L}^\circ(M_r))$, so *every* slope on the companion torus, including the rational longitude, would have to be an interior L-space slope of M_r. The source states that both pieces are Floer simple as a consequence of (5.1); this uses the fact that in the boundary-incompressible manifold setting the interior L-space slope set cannot equal all of $\mathbb{Q}P^1$ (the rational longitude is excluded for rational homology solid tori). Hence $\mathcal{L}^\circ(X_C) \neq \emptyset$. Any slope in it yields a nontrivial L-space surgery on C, since longitude surgery is not an L-space (Lemma 4.1). Thus C is an L-space knot. □

An even shorter argument uses that $\mathcal{L}^\circ(M_r)$ cannot be all slopes because its rational longitude is not an L-space slope. The essential input is not merely that a union covers the circle but that **neither piece can cover it by itself**.

5.3. Second conclusion: the pattern knot is an L-space knot

PROOF Complete deduction from the gluing theorem and filling-commutation homeomorphism By Lemma 4.1, $\lambda_C \notin \mathcal{L}(X_C)$; in particular $\lambda_C \notin \mathcal{L}^\circ(X_C)$. Equation (5.1) forces

$$\boxed{\lambda_C \in h(\mathcal{L}^\circ(M_r)).}$$

Applying h^{-1} and using (1.1), we conclude

$$\boxed{\alpha_P = h^{-1}(\lambda_C) \in \mathcal{L}^\circ(M_r).}$$

Thus $M_r(\alpha_P)$ is an L-space. Equation (3.3) identifies this with $S^3_r(P(U))$. Since $r = p/q$ was chosen with $p \neq 0$, the surgery is nontrivial and produces a rational homology sphere. Therefore the pattern knot $P(U)$ admits a nontrivial L-space surgery, and is an L-space knot. □

The two conclusions yield the desired implication:

$$\boxed{P(C) \text{ L-space knot} \Longrightarrow C \text{ and } P(U) \text{ L-space knots.}}$$

Source versus derived work: Theorem 1.15 and the argument (5.1)–(5.3) are from HRW23, §7.8. The explicit diagram of W with two filling tori, proof of Lemma 4.1, and commuting-fillings explanation are expanded deductions in this lecture. The persistence-of-incompressibility step has been marked as an external topological input rather than silently claimed proved.

6. The 2022 companion: why immersed curves enter

The companion paper is not ancillary to the subject: it explains how the knot invariant underlying the satellite application is visible in the punctured torus. For X_C one has

$$CFK^-(C) \longrightarrow \widehat{CFD}(X_C) \longleftrightarrow \mathcal{C}(X_C).$$

Here the first construction is the LOT knot-complement algorithm, the equivalence with curves is HRW23, Theorem 1.1, and the determination of curves by CFK^- is HRW22, Theorem 11. Do not reverse (6.1) as an unqualified recovery of the **full** CFK^- differential by counting all two-basepoint bigons: the companion §4.3 establishes the associated-graded construction and warns that naive unrestricted bigon counts may fail to square to zero.

6.1. Test arcs versus closed slope circles

For a slope β on ∂M, the ordinary filling $M(\beta)$ is tested by a **closed** embedded slope curve in $T_M = \partial M \setminus \{z\}$:

$$\widehat{HF}(M(\beta)) \cong HF(\mathcal{C}(M), L_\beta^{\text{closed}}).$$

To recover knot Floer homology of the *filling core* $K_\beta \subset M(\beta)$, use instead the **proper noncompact** Lagrangian ending at the puncture:

$$\widehat{HFK}(M(\beta), K_\beta) \cong HF(\mathcal{C}(M), L_\beta^{\text{proper}}),$$

with the relative Spin^c lift specified in HRW22, Proposition 12. These are different pairings. The satellite proof uses ordinary **closed-manifold L-space surgeries**, hence (6.2), not (6.3) directly.

6.2. Floer simplicity and tangent slopes

HRW23, Theorem 7.17 identifies interior L-space slopes with directions avoiding essential singular tangencies of tight immersed curves. If M is Floer simple, HRW22, Lemma 66 (published numbering), shows that an appropriate lifted representative is a graph in coordinates adapted to an interior L-space slope. The torsion values at the lifted punctures are then 0 or 1 according to their side of the graph (HRW22, Proposition 62), and the black–white differences reconstruct the interior interval (HRW22, Theorem 69).

Thus the application to satellites uses more than a formal statement about surgery. It rests on

$$\text{bordered modules} \to \text{graded immersed curves} \to \text{tangent slopes} \to \text{L-space interval} \to \text{gluing}.$$

A geometric caveat: (6.4) classifies **interior** slopes. Endpoint membership must be checked separately; an essential tangent direction may be an endpoint even when the corresponding endpoint filling itself is an L-space.

7. Worked calculation I: a positive trefoil cable

Let $C = T(2,3)$ be the right-handed trefoil and let $P \subset V$ be the $(2,3)$ torus-cable pattern. Then $P(U) = T(2,3)$, and the winding number is $w = 2$. With the usual cabling convention, $K = P(C) = C_{2,3}$. The standard cable-genus formula gives

$$g(K) = 2g(C) + \frac{(2-1)(3-1)}{2} = 2 + 1 = 3.$$

For the right-handed trefoil, the normalized Alexander polynomial is

$$\Delta_C(t) = t^{-1} - 1 + t.$$

The classical satellite Alexander polynomial formula yields (up to a unit, with symmetric normalization)

$$\begin{aligned}\Delta_K(t) &= \Delta_{P(U)}(t)\Delta_C(t^2)\\ &= (t^{-1} - 1 + t)(t^{-2} - 1 + t^2)\\ &= \boxed{t^{-3} - t^{-2} + 1 - t^2 + t^3}.\end{aligned}$$

PROOF Complete polynomial multiplication The nine terms are $t^{-3} - t^{-1} + t$, $-t^{-2} + 1 - t^2$, and $t^{-1} - t + t^3$. Cancel $-t^{-1} + t^{-1} = 0$ and $t - t = 0$ to get (7.3). Substitution $t = 1$ gives $1-1+1-1+1 = 1$, as required.

For a positive L-space knot in S^3 of genus g, the nontrivial rational L-space surgery interval (with the meridian at ∞) begins at $2g - 1$. Therefore the predicted positive threshold for this cable is

$$2g(K) - 1 = 5.$$

The actual **sufficiency** that this particular $(2,3)$-cable is an L-space knot uses the cabling criterion of Jennifer Hom, *A note on cabling and L-space surgeries*: for positive (p,q)-cables, if the companion is a positive L-space knot, then the cable is an L-space knot precisely in the range $q/p \geq 2g(C) - 1$. Here $3/2 \geq 1$, so the criterion applies. Combining it with the genus formula yields, for example, an L-space surgery with slope $r = 6$ and all sufficiently large positive integral slopes.

WARNING Separate the sources The statement *"if the satellite is L-space, then both constituents are"* is HRW23, Theorem 1.15. The converse *for the special cabling pattern with a numerical twisting bound* is Hom's cabling theorem, **not** a converse proved by HRW23, Theorem 1.15.

8. Worked calculation II: necessity is not sufficiency

Retain $C = T(2,3)$ but take the $(2,1)$ cable pattern. The pattern knot $P(U) = T(2,1)$ is an unknot and hence admits L-space fillings; the companion is again the trefoil, an L-space knot. Nevertheless,

$$\frac{q}{p} = \frac{1}{2} < 2g(C) - 1 = 1.$$

By the verified cabling criterion, $C_{2,1}$ has **no positive** L-space surgery. To exclude negative L-space surgery as well, use Hom, *Bordered Heegaard Floer homology and the tau-invariant of cable knots*, Theorem 1. Since the right-handed trefoil has $\tau(C) = \epsilon(C) = 1$, the cable formula gives

$$\tau(C_{2,1}) = 2\tau(C) + \frac{(2-1)(1-1)}{2} = 2, \qquad g(C_{2,1}) = 2.$$

A negative L-space knot has $\tau = -g$ by mirroring a positive L-space knot; here $\tau = 2 \neq -2$. Thus the cable has neither positive nor negative L-space surgery and is **not** an L-space knot. This is an explicit illustration of the logical failure of the naive converse:

$$\boxed{(P(U), C \text{ both L-space knots}) \not\Rightarrow P(C) \text{ an L-space knot.}}$$

Its Alexander polynomial is still a useful exercise: since $P(U)$ is the unknot,

$$\Delta_{C_{2,1}}(t) = \Delta_C(t^2) = t^{-2} - 1 + t^2.$$

The coefficients in (8.3) do not disqualify L-space knot status by themselves. The surgery interval/cabling criterion carries strictly stronger information.

9. Worked calculation III: a non-L-space companion

Let $C = 4_1$ be the figure-eight knot and let P be the positive $(2,3)$ cable pattern, so $P(U)$ is the right-handed trefoil. HRW23, Theorem 1.15 immediately shows $P(C)$ is **not** an L-space knot, because 4_1 admits no nontrivial L-space surgeries.

The Alexander polynomial provides an independent obstruction **in this example**. With symmetric normalization,

$$\Delta_{4_1}(t) = -t^{-1} + 3 - t, \qquad \Delta_{T(2,3)}(t) = t^{-1} - 1 + t.$$

Thus

$$\begin{aligned}\Delta_{P(4_1)}(t) &= (t^{-1} - 1 + t)(-t^{-2} + 3 - t^2) \\ &= \boxed{-t^{-3} + t^{-2} + 2t^{-1} - 3 + 2t + t^2 - t^3}.\end{aligned}$$

In the L-space-knot Alexander polynomial of a knot in S^3, nonzero coefficients alternate between $+1$ and -1; in particular, the coefficient -3 is impossible. Thus (9.1) independently excludes L-space knot status. Yet this method is **not a general substitute** for Theorem 1.15: satellite Alexander polynomials can lose filtered information by cancellation, and an admissible alternating polynomial does not force a knot to be an L-space knot. Theorem 1.15 rules out all such satellite cases geometrically, without requiring a visible bad coefficient.

10. Convention and logical audit

WARNING Five distinctions for an oral examination **(i)** $P \subset V$ versus $P(U) \subset S^3$. The theorem concerns the latter as a knot in S^3.
(ii) T_{out} versus T_{in}. First surger K on the former to obtain M_r with *one* torus boundary; then use the latter in the gluing theorem.
(iii) $\alpha_P = \ell_V$ versus a *meridian of* V. The correct inner filling slope is the meridian of the **complementary solid torus**.
(iv) $\mathcal{L}^\circ$ versus $\mathcal{L}$; Theorem 1.14 is an **interior** covering criterion.
(v) Pattern-and-companion necessity versus an unrestricted converse. A converse needs extra pattern hypotheses or twisting bounds.

10.1. What if the satellite torus compresses after surgery?

A gluing decomposition of a knot exterior does not automatically remain a boundary-incompressible decomposition of every surgery manifold. We explicitly choose a sufficiently generic L-space surgery preserving the essential torus. Applying Theorem 1.14 to a compressed or solid-torus side without checking its hypotheses would create a genuine proof gap. The one-page source proof chooses a *toroidal L-space surgery* at the outset, and so do we.

10.2. Orientation and chirality

An L-space knot can admit positive or negative L-space surgeries depending on orientation/mirroring conventions. A right-handed trefoil is positive, with L-space surgery interval $[1, \infty]$, while the mirror has the corresponding negative interval. Do not equate the sign of a coordinate vector with a geometric Dehn twist without specifying the boundary orientation.

10.3. Homology is necessary but not sufficient

For a knot exterior, $H_1(S^3_{p/q}(K)) \cong \mathbb{Z}/|p|$ whenever $p \neq 0$. Hence the **order of first homology is independent of the knot**, but being an L-space depends on the Floer differential. The equality $|H_1(Y)| = \dim \widehat{HF}(Y)$ is a strong additional condition.

11. Proof checkpoints

- ☐ Explain $X_K = X_C \cup_h W$ and name both torus boundary components of W.
- ☐ Draw the *two-stage* filling $M_r = W(r)$ and $Y_r = X_C \cup_h M_r$.
- ☐ Prove $M_r(\alpha_P) \cong S^3_r(P(U))$ without swapping μ_V and ℓ_V.
- ☐ Prove λ_C is not an L-space filling slope by a complete homology computation.
- ☐ State the boundary-incompressibility hypothesis of HRW23, Theorem 1.14.
- ☐ Identify the external topology input selecting a toroidal L-space surgery.
- ☐ Infer separately that C is an L-space knot and that $P(U)$ is an L-space knot.
- ☐ Explain why HRW22's proper test arc computes HFK of the filling core, not closed HF.
- ☐ Calculate the Alexander polynomial of the $(2, 3)$-cable of the trefoil.
- ☐ Explain why both constituents being L-space knots is not sufficient.

12. Exercises with hints

EXERCISE 1. Exact surgery homology For an arbitrary knot $J \subset S^3$ and coprime integers p, q, use Mayer–Vietoris to prove $H_1(S^3_{p/q}(J)) \cong \mathbb{Z}/|p|$ for $p \neq 0$, and $H_1(S^3_0(J)) \cong \mathbb{Z}$. State exactly where null-homology of λ_J is used.
Hint: Write down generators from X_J and the surgery solid torus, then eliminate the latter.

EXERCISE 2. The commuting-fillings diagram Starting with $W = V \setminus \nu P$, draw both boundary tori and prove $W(r)(\alpha_P) \cong S^3_r(P(U))$. Explain why filling T_{in} along the **meridian** μ_V is not the required operation.
Hint: Identify the meridian of $S^3 \setminus \operatorname{int} V$.

EXERCISE 3. The exact use of gluing Assume $Y_r = X_C \cup_h M_r$ is an L-space with both pieces boundary incompressible. Prove C is an L-space knot and $\alpha_P \in \mathcal{L}^\circ(M_r)$, carefully justifying why $\lambda_C \notin \mathcal{L}(X_C)$. Identify what *additional* fact you need to conclude that both interiors are nonempty.
Hint: Use the rational longitudes and (2.1), not just a set-theoretic union.

EXERCISE 4. Cable Alexander polynomial For $C = T(2, 3)$ compute $\Delta_{C_{2,3}}(t)$ from the satellite formula; verify symmetry and $\Delta(1) = 1$. Compute $g(C_{2,3})$, $2g - 1$, and an explicit positive integer L-space surgery slope, using the cabling theorem.
Hint: $g(C_{p,q}) = pg(C) + (p-1)(q-1)/2$ for positive coprime p, q.

EXERCISE 5. Necessity versus sufficiency Show that the pattern knot of the $(2, 1)$ cable is an unknot; check the inequality $1/2 < 2g(T(2, 3)) - 1$. Why do the two constituents satisfy HRW23's necessary conclusion while the satellite fails to be an L-space knot?
Hint: Apply Hom's **if and only if** cabling criterion in the stated positive setting.

EXERCISE 6. The companion test arc State HRW22, Proposition 12. Contrast the proper slope arc used for $\widehat{HFK}(K_\beta)$ with the closed slope circle used for $\widehat{HF}(M(\beta))$. Why is the Alexander-

grading lift needed in the former?
Hint: Recall Relative Spin-C Structures and the grading cylinder.

EXERCISE 7. Figure-eight obstruction Compute $\Delta_{T(2,3)(4_1)}(t)$ for winding number 2. Verify the coefficient -3. Explain both the polynomial obstruction in this example and the stronger general geometric obstruction from Theorem 1.15.
Hint: Use $\Delta_{4_1}(t) = -t^{-1} + 3 - t$.

EXERCISE 8. Hypothesis audit and attempted converse Suppose only that $P(U)$ and C are L-space knots. List the statements in the proof of Theorem 1.15 that **cannot** be reversed to deduce that $P(C)$ is an L-space knot. Include a concrete (p, q) cabling example; identify the extra threshold in Hom's theorem.
Hint: L-space slopes of the filled pattern exterior M_r depend on r and the pattern, not only on the knot type $P(U)$.

13. Solutions

Solution 1

In $H_1(X_J) \cong \mathbb{Z}\langle\mu_J\rangle$, the preferred longitude vanishes. The surgery meridian is $p\mu_J + q\lambda_J$, so its image imposes $p\mu_J = 0$. The Mayer–Vietoris quotient has no other free generator after eliminating the core generator, hence $H_1 \cong \mathbb{Z}/p\mathbb{Z}$ for $p \neq 0$ and $\mathbb{Z}$ for $p = 0$. The case $p = 0$ has $b_1 = 1$ and cannot be an L-space.

Solution 2

Write $W(r) = W \cup_{T_{\text{out}}} D^2 \times S^1$. The complementary solid torus to unknotted V has meridian $\ell_V = \alpha_P$, so $W \cup_{T_{\text{in}}} (D^2 \times S^1)_{\alpha_P} \cong X_{P(U)}$. Since the fillings occur along disjoint tori, their order may be exchanged, yielding $W(r)(\alpha_P) \cong S^3_r(P(U))$. Filling the inner boundary along μ_V would correspond to another gluing and is not this S^3 placement.

Solution 3

Equation (2.1) gives $\mathcal{L}^\circ(X_C) \cup h(\mathcal{L}^\circ(M_r)) = \mathbb{Q}P^1$. The rational-longitude filling of **each** rational homology solid torus is not an L-space (its b_1 is positive), so neither interior can be the whole slope circle. Therefore both are nonempty, making C Floer simple and hence an L-space knot. Since $\lambda_C \notin \mathcal{L}(X_C)$, the covering equation forces $\lambda_C \in h(\mathcal{L}^\circ(M_r))$, equivalently $\alpha_P \in \mathcal{L}^\circ(M_r)$. Then (3.3) supplies a nontrivial L-space surgery on $P(U)$.

Solution 4

$$\begin{aligned}\Delta(t) &= (t^{-1} - 1 + t)(t^{-2} - 1 + t^2) \\ &= t^{-3} - t^{-2} + 1 - t^2 + t^3.\end{aligned}$$

It is symmetric and evaluates to 1 at $t = 1$. The genus is $g = 3$, so the positive L-space surgery threshold is $2g - 1 = 5$. Hom's criterion applies because $3/2 \geq 1$, so the positive integer slope 6 is an example of an L-space surgery.

Solution 5

The $(2, 1)$ torus knot is an unknot. It is an L-space knot, and so is the right-handed trefoil. But $q/p = 1/2 < 2g(C) - 1 = 1$. Hom's cabling theorem rules out positive L-space surgery. Hom's τ-cabling formula gives $\tau(C_{2,1}) = 2 = g(C_{2,1})$, ruling out negative L-space surgery (which would require $\tau = -g$). Thus the cable is not an L-space knot. HRW23, Theorem 1.15 is a one-way implication and cannot be reversed in general.

Solution 6

HRW22, Proposition 12 identifies each relative Spin^c group $\widehat{HFK}(K_\beta, \bar{\mathfrak{s}})$ with the intersection Floer homology of the graded immersed curve and a chosen lift of the **proper noncompact** test arc $L_{\beta,\bar{\mathfrak{s}}}$. By contrast, the closed circle of slope β computes $\widehat{HF}$ of the filling. A choice of lift records the affine relative Spin^c structure, and its vertical placement determines the Alexander grading after normalization.

Solution 7

$$\Delta(t) = (t^{-1} - 1 + t)(-t^{-2} + 3 - t^2) = -t^{-3} + t^{-2} + 2t^{-1} - 3 + 2t + t^2 - t^3.$$

Its constant coefficient is -3. A knot in S^3 admitting an L-space surgery cannot have such an Alexander coefficient. Theorem 1.15 additionally rules out **any** satellite L-space knot with the figure-eight knot as companion, even in examples where the Alexander coefficients happen not to reveal an obstruction.

Solution 8

Knowledge that C and $P(U)$ individually have L-space fillings does not control whether a specific $M_r = W(r)$ has L-space slopes compatible with those of X_C. In particular, the covering condition (2.1) can fail. The $(2, 1)$ cable of the right-handed trefoil is a counterexample to the unrestricted converse; the cabling-specific sufficiency additionally requires $q/p \geq 2g(C) - 1$.

14. Oral examination

QUESTION Q1. State HRW23, Theorem 1.15 with every relevant distinction. **Model answer:** For a satellite $K = P(C)$ in S^3, if K admits a nontrivial L-space surgery, then the companion C and the pattern knot $P(U)$ in S^3 each admit a nontrivial L-space surgery. The conclusion is necessary; the theorem is not a converse.

QUESTION Q2. Where exactly is the gluing theorem applied? **Model answer:** Apply HRW23, Theorem 1.14 to the *closed surgery manifold* $Y_r = X_C \cup_h M_r$ after filling the **outer** boundary of the two-torus pattern exterior. This leaves M_r with one inner torus boundary and lets both pieces be boundary incompressible. Applying the theorem directly to W would be invalid.

QUESTION Q3. Why does the companion have an L-space surgery? **Model answer:** The gluing theorem says the two interior L-space intervals cover the slope circle. Neither is the entire circle because each rational homology solid torus has a non-L-space rational-longitude filling. Hence $\mathcal{L}^\circ(X_C) \neq \emptyset$. Any such slope is nonlongitudinal and gives an L-space surgery of C.

QUESTION Q4. Why is the pattern filling slope $h^{-1}(\lambda_C)$? **Model answer:** The zero-framed embedding identifies the longitude ℓ_V of the standard unknotted solid torus with the companion's Seifert longitude. Its inverse image $\alpha_P = \ell_V$ is the meridian of the solid torus complementary to V in S^3. Therefore inner filling along α_P recovers the pattern knot exterior in S^3.

QUESTION Q5. What does the 2022 companion add conceptually? **Model answer:** HRW22 explains how the knot's CFK^- determines the immersed-curve invariant, how proper test arcs compute knot Floer homology of filling cores, and how Spin^c refinements and Turaev torsion encode filling geometry. HRW23 supplies the immersed-curve gluing theorem used to prove the satellite result.

QUESTION Q6. Is an L-space-pattern knot together with an L-space companion sufficient? **Model answer:** No. The $(2, 1)$ cable pattern is an unknot and the right-handed trefoil is an L-space companion, but the cable has no positive L-space surgery since $q/p = 1/2 < 2g(C) - 1 = 1$, and no negative L-space surgery since $\tau = 2$ whereas negative L-space knots have $\tau = -g = -2$. For positive cables the extra numerical sufficiency criterion is Hom's cabling theorem.

QUESTION Q7. Why is the Alexander polynomial calculation insufficient in general? **Model answer:** For an L-space knot, the Alexander polynomial has a restricted alternating ± 1 coefficient structure, so a bad coefficient excludes some examples. But the Alexander polynomial is a decategorified Euler characteristic and cannot certify L-space surgery from good coefficients. HRW's theorem instead uses the entire Floer-simple gluing condition.

QUESTION Q8. What is the hardest hidden hypothesis in the one-page source proof? **Model answer:** One needs a toroidal L-space surgery preserving the essential companion torus, so both sides of the surgery decomposition have incompressible torus boundary. The main paper chooses such a surgery at the start of §7.8; it does not reprove general essential-surface persistence there. One cannot apply Theorem 1.14 at an exceptional compressing surgery without this check.

15. Four-hour-fifty-minute plan

Time	Primary reading / task	Checkpoint
40 min	HRW23, introduction Thms 1.14–1.15; §7.8 printed p. 84	State both theorems and locate both boundary tori.
50 min	Work through $X_K = X_C \cup W$, $M_r = W(r)$, and the inner-filling diagram	Prove equation (3.3) without changing a longitude to a meridian.
50 min	Reconstruct the satellite proof from §7.8 and Theorem 1.14	Identify exactly where the zero longitude and incompressibility enter.
45 min	HRW22, §4 (Theorem 11; Proposition 12), §5.3 (Lemma 66; Theorem 69)	Explain curve-pairing, slope interval, and relative grading roles.
55 min	Compute the two cabling examples, genus, Alexander polynomials, and thresholds	Separate an external cabling theorem from HRW necessity.
50 min	Exercises 1–3, 7–8 and eight oral questions	Defend the proof hypotheses and every gluing map.

Total: 290 minutes = 4 hours 50 minutes.

16. Verified references

[HRW23] J. Hanselman, J. Rasmussen, L. Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, arXiv:1604.03466v3, April 25, 2023. Exact verified source: Theorems **1.14, 1.15**, §**7.8** (arXiv printed p. **84**, PDF page 84 counting from 1), and Conjecture **7.21**. Later journal version: *J. Amer. Math. Soc.* **37** (2024), 391–498, doi:10.1090/jams/1029. The journal pagination is not substituted for the arXiv pagination.

[HRW22] J. Hanselman, J. Rasmussen, L. Watson, *Heegaard Floer homology for manifolds with torus boundary: properties and examples*, *Proc. London Math. Soc.* **125** (2022), 879–967, doi:10.1112/plms.12473. Verified cross-version anchors: Theorem **11** and Proposition **12** in the introduction; §**4** on knot Floer homology; §**5.3** on L-space intervals. The preprint arXiv:1810.10355v1 differs by one in numbering for several later results: the associated-graded theorem is **51** in v1 and **52** in the published version. We do not use an unverified published page number for that theorem.

[HLV14] J. Hom, T. Lidman, F. Vafaee, *Berge–Gabai knots and L-space satellite operations*, *Algebr. Geom. Topol.* **14** (2014), 3745–3763, doi:10.2140/agt.2014.14.3745. HRW23 identifies **Conjecture 1.7** there as the conjecture resolved by its Theorem 1.15.

[Hom11] J. Hom, *A note on cabling and L-space surgeries*, *Algebr. Geom. Topol.* **11** (2011), 219–223, doi:10.2140/agt.2011.11.219. We use its **positive cabling iff criterion** only for the explicit (p, q) cable calculations; this is a supporting theorem, not attributed to HRW.

[Hom14, supporting only] J. Hom, *Bordered Heegaard Floer homology and the tau-invariant of cable knots*, *J. Topology* **7** (2014), 287–326, doi:10.1112/jtopol/jtt030. Verified arXiv:1202.1463v2, Theorem 1: for $\epsilon(C) = 1$, $\tau(C_{p,q}) = p\tau(C) + (p-1)(q-1)/2$. Used only to rule out negative L-space surgeries in the counterexample.

[LOT18, supporting only] R. Lipshitz, P. Ozsvath, D. Thurston, *Bordered Heegaard Floer Homology*, *Memoirs AMS* **254** (2018), no. 1216, doi:10.1090/memo/1216; Chapters 8–11 supply the algebraic framework used by HRW, not a third primary paper for this lecture.