Day 18: Injectivity, Test Curves, and Local-System Rigidity

HRW 21-Day Rigorous Qualifying Examination

9 November 2026

Monday, November 9, 2026. A 4-hour-50-minute rigorous English qualifying-exam lecture. **Main reading:** HRW23, §§4.1–4.3, Theorem 4.11, Lemma 4.14, Proposition 4.15. **Companion reading:** HRW22, §2.1 (gradings and phantom arrows), §3 (elliptic symmetry). **PDF attachment:** HRW_Day18.pdf. Prior notes: Day 17, Day 9.

1. Exact classification statement and logical map

The previous lecture established the existence half: a reduced **extendable** type D structure can be simplified into unobstructed immersed curves decorated by local systems. Existence alone is insufficient. Two different curve pictures could, a priori, encode the *same* algebraic object. Today's target is to exclude this ambiguity.

Let $T = T^2 \setminus \{z\}$ be the marked punctured torus with parametrizing arcs α, β, and let $\mathbb{F} = \mathbb{F}_2$. Write $\mathcal{D}$ for homotopy-equivalence classes of extendable type D structures over the torus algebra, and $\mathcal{C}$ for the equivalence classes of **finite unobstructed decorated immersed multicurves** as defined in HRW23, §4.2. The equivalence is more permissive than regular homotopy of a fixed list of components: it permits orientation reversal with inverse monodromy, direct sum on equal primitive curves, and primitive reduction of multiple covers.

> **THEOREM** Source theorem — HRW23, Theorem 4.11 (printed p. 46) There is a bijection $f : \mathcal{D} \to \mathcal{C}$ with inverse $g : \mathcal{C} \to \mathcal{D}$, compatible with the algebraic box-tensor and the geometric immersed-Floer pairings (with the paper's type-A/type-D reflection convention).

This is an **ungraded algebraic classification**. For an actual manifold M, HRW23, Theorems 1.1 and 1.4 subsequently provide parametrization independence and extendability of $\widehat{CFD}(M)$. These are logically different statements. More precise grading information belongs to HRW22, Theorem 5.

The proof consists of four layers:

$$\begin{gathered}\text{curve } (\gamma, V, A) \xrightarrow{g} \text{ extendable type D structure,}\\ \text{pairing compatibility (HRW23, Proposition 4.13),}\\ \text{recover underlying conjugacy classes (Lemma 4.14),}\\ \text{recover similarity classes of } A \text{ (Proposition 4.15).}\end{gathered}$$

Oral-exam standard: state whether a claim is about modules, immersed curves, local systems, or geometric realizability. Even a perfect classification of extendable modules is **not** a theorem saying every such module arises from a three-manifold.

2. Unobstructed curves and admissible Floer tests

> **THEOREM** Source definitions — HRW23, Definitions 4.1, 4.2, 4.7 An immersed loop $\gamma : S^1 \looparrowright T$ is **unobstructed** if the bi-infinite lift $\tilde{\gamma} : \mathbb{R} \to \widetilde{T}$ is embedded in the universal cover. Two loops are **commensurable** if nonzero powers represent the same free homotopy class, equivalently if the corresponding cyclically reduced free-group words have a common primitive root up to inversion and conjugacy. For a commensurable pair, admissibility is expressed by the associated periodic annular domain having both positive and negative multiplicities.

Here $\pi_1(T) \cong F(\alpha, \beta)$ is **nonabelian**. Abelianized slope data in $H_1(T) \cong \mathbb{Z}^2$ need not classify immersed free homotopy classes. By contrast, an embedded primitive essential curve in an *unpunctured torus* is classified by its primitive slope. Confusing these settings is a common oral-exam error.

Example (derived). Compare the cyclically reduced words

$$w_1 = \alpha, \qquad w_2 = \alpha^2\beta\alpha^{-1}\beta^{-1}.$$

Both have abelianization $(1, 0)$, yet their cyclically reduced lengths are respectively 1 and 5, so they are not conjugate in the free group. The associated reduced train-track paths have different free homotopy types. This is why HRW must probe *words*, not merely the two homology coordinates. (This example is not claimed to represent two particular manifold invariants.)

If γ, γ' are incommensurable unobstructed loops, geodesic representatives have finitely many transverse intersections and no immersed bigons between distinct intersection points. If they are commensurable, bigon homotopy classes may form a $\mathbb{Z}$-torsor, so an annular periodic domain must be treated. HRW23, Lemma 4.4 describes the dichotomy; Definition 4.7 fixes admissibility; Theorem 4.9 establishes invariance under admissible positioning.

> **WARNING** A minimal drawing need not be admissible Two parallel circles can have geometric intersection number zero but nonzero Floer homology. Naively declaring their Floer group zero because a drawing has no intersections is incorrect. Use an admissible perturbation, or the local-system cochain model below.

3. Primitive reduction and its monodromy trap

Let (γ, k, A) denote an oriented unobstructed loop, a rank-k local system, and a monodromy $A \in GL_k(\mathbb{F})$ up to similarity. HRW23, Lemma 4.8 and §4.2 impose the following operations:

1. Replace γ by a freely homotopic loop.
2. Reverse orientation, replacing A by A^{-1} (and accounting for dualization when constructing the opposite-oriented Floer test).
3. Merge identical underlying oriented curves by replacing A_1, A_2 with $A_1 \oplus A_2$.
4. If γ traverses a primitive δ exactly m times, replace (γ, k, A) by $(\delta, mk, A_{\text{ind}})$, where A_{ind} is the cyclic block matrix with identities on the subdiagonal and A in the top-right corner.

In block form,

$$A_{\text{ind}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & A \\ I_k & 0 & \cdots & 0 & 0 \\ 0 & I_k & \ddots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_k & 0 \end{pmatrix}.$$

Worked double-cover test. The double traversal of a primitive δ with rank-one monodromy 1 becomes a rank-two local system on δ with

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S^2 = I, \qquad (S + I)^2 = 0, \qquad S \neq I.$$

Over $\mathbb{F}_2$, $\chi_S(t) = (t + 1)^2$ and the minimal polynomial is $(t + 1)^2$. The direct sum of *two trivial rank-one local systems* has monodromy I_2, whose minimal polynomial is just $t + 1$. Therefore the multiple cover is **not** equivalent to two trivial local systems. This is precisely the information that primitive reduction must preserve.

> **PROOF** Derived consequence Similarity preserves minimal polynomials; since $m_S(t) = (t + 1)^2 \neq t + 1 = m_{I_2}(t)$, no invertible basis change can identify S with I_2.

4. What nonparallel pairings can detect

> **THEOREM** Source formula — HRW23, Corollary 4.10 (printed p. 45) For primitive unobstructed curves (γ, k, A), (γ', k', B), if the underlying curves are incommensurable, then
>
> $$\dim HF((\gamma, k, A), (\gamma', k', B)) = kk'\, i(\gamma, \gamma').$$
>
> For opposite-oriented representatives of the same primitive curve, the rank includes a $2 \dim \ker(I \otimes I + A \otimes B)$ contribution in addition to twice the product of the ranks times the minimal self-intersection number $i(\gamma)$.

For an embedded primitive loop, $i(\gamma) = 0$. The formula then becomes

$$\boxed{\dim HF((\gamma, k, A), (\gamma^{-1}, k', B)) = 2 \dim \ker(I_{kk'} + A \otimes B).} \tag{4.1}$$

The orientation/dualization convention matters: for a **self-pairing** of one geometric local system with itself, the matrix on the oppositely oriented second factor is $(A^{-1})^T$. Under the identification $V \otimes V^* \cong \mathrm{End}(V)$ the operator becomes

$$\Delta(X) = X + AXA^{-1}. \tag{4.2}$$

Its kernel is the centralizer $\mathrm{Cent}(A) = \{X : AX = XA\}$; in particular, I belongs to the kernel. Thus the self-pairing never vanishes.

Example. If two embedded straight primitive loops have slopes $(1, 0)$ and $(3, 7)$ and their local-system ranks are 2 and 3, then

$$i = \left|\det \begin{pmatrix} 1 & 3 \\ 0 & 7 \end{pmatrix}\right| = 7, \qquad \dim HF = 2 \cdot 3 \cdot 7 = 42.$$

The monodromy did not enter. Thus merely testing against **nonparallel rational slope circles**, even all of them, cannot distinguish nonisomorphic monodromies of fixed rank. Parallel local-system tests are indispensable.

A cellular model for the parallel pairing (derived proof)

Let V, W carry invertible monodromies A, C. The degree-zero and degree-one cellular cochain groups of a circle with coefficients in $\mathrm{Hom}(W, V)$ are both $\mathrm{Hom}(W, V)$. The differential is, up to invertible pre/post-composition,

$$d(X) = AX - XC.$$

Over $\mathbb{F}_2$ this is $AX + XC$. Therefore

$$H^0 \cong \{X : AX = XC\}, \qquad \dim H^1 = \dim H^0$$

by rank-nullity, since the chain groups have equal dimension. Thus the total rank is

$$\boxed{2 \dim \mathrm{Hom}_{\mathbb{F}[t,t^{-1}]}(W_C, V_A).} \tag{4.3}$$

This is the algebraic form of the commensurable Floer calculation. **The identification of this cochain model with immersed Floer homology is HRW's geometric theorem**; the cochain rank calculation itself is proved here.

5. Detecting cyclic words with nonclosed test tracks

The first half of injectivity must recover all underlying free homotopy classes and their total ranks *before* trying to recover monodromy.

Represent a primitive underlying curve by a cyclically reduced word in $\alpha^{\pm 1}, \beta^{\pm 1}$. For a finite word I, define $n_I(\vartheta)$ to be the weighted number of occurrences of I or I^{-1} in the cyclic words of the curve-like train track ϑ, counting the local-system rank as multiplicity. Count occurrences cyclically and with overlaps.

HRW construct a **dual open test track** $\vartheta_{\bar{I}}$. It is a nonclosed immersed train track, generally only *weakly valid*, not extendable: HRW23, §4.3, Figure 40 and the passage preceding Lemma 4.14, explicitly distinguish these. Such objects are permitted as algebraic probes in the chain-level test pairing; they are not alleged to be three-manifold invariants.

> **THEOREM** Source theorem — HRW23, Lemma 4.14 (printed pp. 49–51) The dimension of $H_*C(\vartheta, \vartheta_{\bar{I}})$, together with the previously determined weighted counts of shorter subword patterns, determines $n_I(\vartheta)$. The proof is an induction using the combinatorics of bigons and zig-zag chains, illustrated by Figure 41.

Fully computed three-letter recurrence

For $I = \alpha\beta^{-1}\alpha$, HRW's Figure 41 produces

$$\dim H_* C(\vartheta, \vartheta_{\bar{I}}) = 2n_\alpha + n_\beta - 2n_{\alpha\beta^{-1}} - 2n_{\beta^{-1}\alpha} + 2n_{\alpha\beta^{-1}\alpha}. \tag{5.1}$$

All n's are evaluated on ϑ. Hence

$$\boxed{n_{\alpha\beta^{-1}\alpha} = \tfrac{1}{2}\left(\dim H_* - 2n_\alpha - n_\beta + 2n_{\alpha\beta^{-1}} + 2n_{\beta^{-1}\alpha}\right).}$$

The division by two here is in the **integers**, not in $\mathbb{F}_2$: $\dim H_*$ and the n_I are ordinary nonnegative integer counts.

For a single cyclic word $w = \alpha\beta^{-1}\alpha$ with rank-one decoration, the counts are $n_\alpha = 2$, $n_\beta = 1$, $n_{\alpha\beta^{-1}} = n_{\beta^{-1}\alpha} = n_{\alpha\beta^{-1}\alpha} = 1$. Equation (5.1) yields

$$\dim H_* = 4 + 1 - 2 - 2 + 2 = 3.$$

With a rank-two decoration all counts double, giving homology dimension 6. The open test does not see the *matrix* of the local system, only its rank: no closed turn around the decorated component is forced by this test.

Proof reconstruction of the general recurrence

The test track has intersections associated with letters of I; its bigons correspond to shorter matching subwords. An intersection can be an initial point of at most two relevant bigons, or a terminal point of at most two; the two roles do not overlap. Hence the complex decomposes into alternating zig-zag chains. Counting homology of these chains and correcting for subchains expresses the result as known shorter-word contributions plus the desired n_I. HRW23, printed pp. 50–51, prove the geometric uniqueness of the relevant bigons via the monotone northwest movement of boundary paths (Figure 11). **This is a proof sketch of the diagrammatic step; the exact local classification is not reproduced.**

Knowing all weighted cyclic subword counts recovers the finite multiset of unoriented cyclically reduced words with multiplicities. One way to see the principle is to use increasingly long repeated patterns: two distinct primitive periodic bi-infinite words cannot agree on arbitrarily long finite segments. For a complete proof in the present setting, the tests and induction are part of HRW23, §4.3, not a consequence of abelianized slopes.

6. Complete algebraic proof that monodromy is detected

We now prove the linear-algebra statement underpinning HRW23, Proposition 4.15. The source **outlines** this proof and leaves detailed rational-canonical-form steps to the reader; here we supply the missing details.

> **THEOREM** Source theorem — HRW23, Proposition 4.15 (printed p. 52) If $A_1, A_2 \in GL_k(\mathbb{F})$ are not similar, then there exists an invertible matrix B (possibly of different size) for which
>
> $$\operatorname{rank}(I \otimes I + A_1 \otimes B) \neq \operatorname{rank}(I \otimes I + A_2 \otimes B).$$

Theorem 6.1 — Complete derived proof via primary modules

Let $R = \mathbb{F}[t, t^{-1}]$, and let V_A be the finite torsion R-module on which t acts as an invertible matrix A. By the structure theorem for finitely generated torsion modules over a PID,

$$V_A \cong \bigoplus_p \bigoplus_{j=1}^{r_p} R/(p^{\lambda_{p,j}}), \tag{6.1}$$

where the $p \in \mathbb{F}[t]$ are monic irreducible polynomials with $p(0) \neq 0$ and the positive integers $\lambda_{p,j}$ are the elementary-divisor block lengths.

For a fixed p and $\ell \geq 1$, let $C_{p,\ell}$ be the invertible companion matrix of p^ℓ; its dimension is $d\ell$, where $d = \deg p$. Set

$$B_{p,\ell} = (C_{p,\ell}^{-1})^T.$$

Identify $\mathbb{F}^k \otimes (\mathbb{F}^{d\ell})^*$ with $\mathrm{Hom}(\mathbb{F}^{d\ell}, \mathbb{F}^k)$ using columns/dual rows. Under this identification,

$$(A \otimes B_{p,\ell})(X) = AXC_{p,\ell}^{-1}.$$

Since the coefficient field has characteristic two,

$$(I + A \otimes B_{p,\ell})X = 0 \iff AX = XC_{p,\ell}.$$

Thus

$$\boxed{\dim\ker(I + A \otimes B_{p,\ell}) = \dim \mathrm{Hom}_R(R/(p^\ell), V_A).} \tag{6.2}$$

For $q \neq p$, the ideals (q^a) and (p^ℓ) are coprime, so

$$\mathrm{Hom}_R(R/(p^\ell), R/(q^a)) = 0.$$

For $q = p$, an R-linear map $R/(p^\ell) \to R/(p^a)$ is determined by the image x of 1 subject to $p^\ell x = 0$. Hence its $\mathbb{F}$-dimension is $d\min(a, \ell)$. Applying this to (6.1) gives

$$\boxed{D_p(\ell; A) := \dim\ker(I + A \otimes B_{p,\ell}) = d\sum_{j=1}^{r_p} \min(\ell, \lambda_{p,j}).} \tag{6.3}$$

Set $D_p(0; A) = 0$. The discrete first difference satisfies

$$\frac{D_p(\ell; A) - D_p(\ell - 1; A)}{d} = \#\{j : \lambda_{p,j} \geq \ell\}.$$

Therefore the multiplicity $b_{p,\ell}$ of blocks of *exact* size ℓ is

$$\boxed{b_{p,\ell} = \frac{2D_p(\ell; A) - D_p(\ell - 1; A) - D_p(\ell + 1; A)}{d}.} \tag{6.4}$$

Consequently, the integer-valued data $D_p(\ell; A)$ for all p, ℓ uniquely reconstruct every elementary divisor and hence the similarity class of A.

If A_1 and A_2 are not similar, some $D_p(\ell; A_i)$ differ. But both tensor-test matrices have the same total dimension $k \cdot d\ell$, and

$$\mathrm{rank}(I + A_i \otimes B_{p,\ell}) = kd\ell - D_p(\ell; A_i).$$

The ranks therefore differ for that one choice of $B = B_{p,\ell}$. This proves Proposition 4.15. □

> **WARNING** Same characteristic and minimal polynomial do not imply similarity The test family above detects the **multiplicities of elementary divisors**, not merely the set of irreducible factors or the largest Jordan block. A proof that only selects the minimal polynomial of A is insufficient without further argument.

7. Explicit matrix computations

Example A: rank-two systems on the same circle

Let

$$A = I_2, \qquad C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Here $\chi_A = (t+1)^2$ and $\chi_C = t^2 + t + 1$. Take the one-dimensional test $B = [1]$. Then

$$\operatorname{rank}(I_2 + A) = 0, \qquad \operatorname{rank}(I_2 + C) = 2.$$

The kernels have dimensions 2 and 0, respectively. By (4.1), the parallel embedded-curve Floer dimensions are 4 and 0. This is an explicit **Floer detection of monodromy**, not merely a polynomial calculation.

Example B: even minimal and characteristic polynomials are insufficient

Over $\mathbb{F}_2$, let $J_n = I_n + N_n$ be a unipotent Jordan block with ones on the superdiagonal. Set

$$A_1 = J_4 \oplus J_2 \oplus J_2, \qquad A_2 = J_4 \oplus J_3 \oplus J_1.$$

Both matrices lie in $GL_8(\mathbb{F}_2)$, have characteristic polynomial $(t+1)^8$, minimal polynomial $(t+1)^4$, and exactly three Jordan blocks; so the trivial rank-one test $B = [1]$ gives the same nullity 3.

Choose $p = t + 1$, $\ell = 2$, so $C = J_2$ and $B = (J_2^{-1})^T$ (over characteristic two $J_2^{-1} = J_2$). Formula (6.3) gives

$$D_p(2; A_1) = \min(2, 4) + 2 + 2 = 6,$$

and

$$D_p(2; A_2) = \min(2, 4) + \min(2, 3) + \min(2, 1) = 5.$$

Both tensor maps act on a 16-dimensional space. Therefore

$$\boxed{\operatorname{rank}(I + A_1 \otimes B) = 10, \qquad \operatorname{rank}(I + A_2 \otimes B) = 11.}$$

The corresponding parallel Floer dimensions are 12 and 10. This is a genuinely stronger test than checking characteristic polynomials, minimal polynomials, or fixed-point dimensions.

Example C: self-Floer detects centralizer size

For an embedded primitive circle with monodromy A, equation (4.2) gives

$$\dim HF((\gamma, A), (\gamma, A)) = 2 \dim \operatorname{Cent}(A).$$

For I_2, the full 2×2 matrix algebra commutes with it, so $\dim \operatorname{Cent}(I_2) = 4$ and the self-Floer rank is 8. For the irreducible 2×2 matrix C in Example A, every commuting matrix is a polynomial in C of degree < 2, so $\dim \operatorname{Cent}(C) = 2$ and the self-Floer rank is 4.

These are abstract local-system examples. No assertion is made that these nontrivial systems arise from an actual torus-boundary three-manifold.

8. The classification proof assembled

Let $g : \mathcal{C} \to \mathcal{D}$ be the curve-to-module construction in HRW23, §4.2.

Well-definedness and pairings (source: Proposition 4.13). Choose parallel strands representing each rank-k system and factor its invertible monodromy into elementary matrices, each represented by crossover

arrows. Lemmas 4.8 and 4.12, together with the chain-level Theorem 2.2 and admissible finger moves, identify the geometric Floer pairing with the algebraic box tensor pairing. These choices do not change the homotopy class of the resulting extendable module.

Surjectivity (source: §4.2, proof of Theorem 4.11). Given an extendable type D module, use Theorem 1.5 to obtain a curve-like train track. Its closed strands yield decorated curves. The lifted strands are embedded in the universal cover because a reduced track has no immediate backtracking in the lifted four-valent tree; hence they are unobstructed. The resulting element of $\mathcal{C}$ maps back to the given module.

Injectivity, stage one. If $g(\Gamma_1) \simeq g(\Gamma_2)$, their box-tensor pairings agree with every permissible test type D module. In particular they agree with the weakly valid nonclosed tracks $\vartheta_{\bar{I}}$. Lemma 4.14 recovers every weighted cyclic subword count. Therefore their primitive underlying closed curves agree as a multiset, weighted by local-system rank.

Injectivity, stage two. Once a primitive curve and its total rank have been isolated, test it against parallel primitive curves equipped with chosen monodromies B. Corollary 4.10 converts the Floer dimensions to the nullities of $I + A \otimes B$. Proposition 4.15, now proved algebraically in §6, shows that these nullities determine the similarity class of A. Hence the local systems agree.

Therefore g is bijective, and the pairings are respected: this is HRW23, Theorem 4.11. **Only after this classification**, HRW23, Proposition 4.16 concludes that an extendable type D module has an extension unique up to homotopy.

> **PROOF** Status of the proof reconstruction The PID/rational-canonical-form separation of monodromies (§6) and the explicit linear-algebra calculations (§7) are complete. The cyclic-word-to-bigons correspondence of Lemma 4.14 and the admissible geometric comparison in Proposition 4.13 are described as **source-dependent proof sketches**, because their local diagrammatic verification is not reproduced in full.

9. The 2022 companion: grading arrows and manifold-only symmetries

The 2022 companion does not redo the ungraded injectivity proof. Instead it shows that the decorated multicurve model can carry additional **grading data**.

> **THEOREM** Source result — HRW22, Theorem 5 and Definition 29 (§2.1) The relative Spin^c and Maslov gradings on the closed-manifold pairing can be encoded geometrically. A set of *grading arrows* (also called **phantom arrows**) joining the components makes the entire curve collection connected for grading comparisons; the arrows are ignored when counting the bigons that define ordinary intersection Floer homology.

For n components, the chosen grading arrows can be taken to be $n-1$ connecting arrows. Their role differs from local-system crossover arrows: a phantom arrow changes the *grading bookkeeping*, not the underlying ungraded type D object or the ordinary Floer differential. Compare Spin-c grading, Maslov grading, and Crossover arrows.

> **THEOREM** Source results — HRW22, Theorem 7 and Proposition 10 (§3) For **actual manifold invariants**, the elliptic involution acts as Spin^c conjugation, and no component is a small peripheral circle linking the puncture. These are additional manifold restrictions, not assertions about all abstract extendable curve objects.

The latter peripheral-circle prohibition uses an enhanced minus-type relation beyond ordinary extendability. It is therefore logically invalid to invoke it while proving Theorem 4.11 for **all extendable modules**. Conversely, the open test tracks $\vartheta_{\bar{I}}$ used in Lemma 4.14 are generally not themselves extendable or manifold-realizable; that does not prevent their use in a homological *test pairing* with the object being classified.

The hierarchy is:

$$\boxed{\text{ungraded curve classification} \subsetneq \text{fully graded curve information} \quad \text{(additional decorations)},}$$

and separately

$$\boxed{\text{manifold-realizable extendable modules} \subset \text{all extendable modules}.}$$

The first display concerns information content, not a literal inclusion of sets. The second is an inclusion of classes; the source supplies examples of abstract objects excluded from the manifold category.

10. Reading assignments and proof checkpoints

Duration	Reading or work	Checkpoint
35 min	HRW23, §4.1, Definitions 4.1–4.7 and Corollary 4.10	State unobstructedness, commensurability and admissibility exactly.
40 min	HRW23, §4.2, Lemma 4.8 and Theorem 4.11	Explain primitive reduction and the pairing-preserving map g.
55 min	HRW23, §4.3, Lemma 4.14, Figures 40–41	Reconstruct the three-letter recurrence and the bigon/zig-zag mechanism.
55 min	HRW23, Proposition 4.15; complete PID proof in §6	Derive the kernel dimensions and the second-difference formula.
35 min	HRW22, §2.1 and §3	Separate grading arrows from local systems; elliptic symmetry from general algebra.
70 min	Written exercises and oral practice	Solve at least Exercises 2, 4, 5, 7, and defend three proof gaps.

Total: 290 minutes = 4 hours 50 minutes.

Proof checkpoints. Be prepared to derive (i) why simple slope pairings do not detect monodromy; (ii) why a nonclosed test can detect a cyclic-word count; (iii) why primitive reduction needs a block permutation matrix; (iv) why $\dim\ker(I + A \otimes B)$ recovers *all* elementary divisors; (v) where geometric admissibility is needed; and (vi) why the 2022 manifold-only theorems cannot replace the 2023 algebraic injectivity proof.

11. Eight qualifying-exam exercises

EXERCISE 1. Definitions and counterexamples Give the precise definitions of unobstructedness, commensurability and admissible position. Compare the cyclically reduced words α and $\alpha^2\beta\alpha^{-1}\beta^{-1}$. Calculate their abelianizations and explain why a slope-only classification fails. **Hint:** Cyclically reduced word length is a conjugacy invariant.

EXERCISE 2. Primitive reduction A rank-one system of monodromy 1 traverses a primitive curve twice. Compute the rank-two induced monodromy, its square, characteristic polynomial and minimal polynomial; distinguish it from two trivial systems by a rank-one parallel Floer test. **Hint:** Compare $\ker(I + S)$ and $\ker(I + I_2)$.

EXERCISE 3. Bigon recurrence For the cyclic word $\alpha\beta^{-1}\alpha$ with rank three, calculate all weighted counts in (5.1) and the predicted homology dimension of the dual open test. Explain why the monodromy matrix is invisible to this particular test. **Hint:** Double counting is not taken modulo two.

EXERCISE 4. The cyclic cochain model Consider the cellular two-term complex on $\mathrm{Hom}(W, V)$ with differential $d(X) = AX - XC$. Prove its total homology dimension is twice the kernel dimension. Interpret that kernel as a space of R-module homomorphisms for $R = \mathbb{F}[t, t^{-1}]$. **Hint:** Rank-nullity.

EXERCISE 5. Complete Proposition 4.15 Prove the cyclic-primary Hom dimension formula from §6, including vanishing for distinct irreducible factors. Deduce equations (6.3)–(6.4) and the rank-separation theorem. **Hint:** A homomorphism is determined by the image of 1.

EXERCISE 6. Same min/characteristic polynomials For $A_1 = J_4 \oplus J_2 \oplus J_2$ and $A_2 = J_4 \oplus J_3 \oplus J_1$, check equal characteristic polynomial, equal minimal polynomial and equal $\dim\ker(I + A_i)$. Then choose $B = (J_2^{-1})^T$ and compute both tensor ranks. **Hint:** Use $\sum \min(2, \lambda_j)$.

EXERCISE 7. Self-pairing and centralizers Prove that the self-pairing of an embedded primitive curve with monodromy A has dimension $2 \dim \mathrm{Cent}(A)$. Evaluate this for I_2 and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Explain why zero geometric intersection does not imply zero Floer homology.

EXERCISE 8. Full classification audit In the proof of HRW23, Theorem 4.11, identify the source for well-definedness, pairing compatibility, surjectivity, underlying-word injectivity and local-system

injectivity. Why may the weakly valid open tests be used? Where must admissible finger moves be invoked? Explain why uniqueness of extension is a **consequence**, not an assumption.

12. Exercise answers and proof hints

Answer 1. The words have exponent-sum vectors $(1,0)$ and $(1,0)$. Both are cyclically reduced, with lengths 1 and 5, so they are not conjugate. Use the definitions in §2 for the remaining terms. An immersed closed curve that lifts without self-intersections is unobstructed; commensurability is a common-power relation in the free group; annular admissibility requires positive and negative periodic-domain multiplicities.

Answer 2. $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $S^2 = I_2$, and $m_S(t) = (t+1)^2$ versus $m_{I_2}(t) = t+1$. Since $\dim\ker(I+S) = 1$ and $\dim\ker(I+I_2) = 2$, the corresponding parallel rank-one Floer tests give dimensions 2 and 4.

Answer 3. The rank-three multiplicity gives $n_\alpha = 6$, $n_\beta = 3$, $n_{\alpha\beta^{-1}} = n_{\beta^{-1}\alpha} = n_{\alpha\beta^{-1}\alpha} = 3$. Thus (5.1) gives $12 + 3 - 6 - 6 + 6 = 9$. Only the number of strands matters because the open test does not force a return that samples the full monodromy.

Answer 4. The source and target of d each have dimension $(\dim V)(\dim W)$. If $r = \operatorname{rank} d$, then $\dim H^0 = k\ell - r$ and $\dim H^1 = k\ell - r$, so the total is $2(k\ell - r) = 2\dim\ker d$. The kernel is precisely the intertwiner space $AX = XC$.

Answer 5. The image of 1 must be killed by p^ℓ. In $R/(p^a)$ the annihilator of p^ℓ has $\mathbb{F}$-dimension $\deg p \min(a,\ell)$. Distinct primary polynomials are coprime, forcing zero. Sum over primary blocks and take successive differences to recover block multiplicities. This proves Proposition 4.15 in full.

Answer 6. Both characteristic polynomials are $(t+1)^8$, both minimal polynomials are $(t+1)^4$, and both have three Jordan blocks, hence $\dim\ker(I + A_i) = 3$. For the rank-two test, nullities are 6 and 5 on a dimension-16 space, so ranks are 10 and 11.

Answer 7. With the opposite-orientation monodromy $(A^{-1})^T$, the kernel equation is $AX = XA$, and H^0, H^1 each have that dimension. For I_2, centralizer dimension 4 gives Floer rank 8. For an irreducible quadratic companion, centralizer dimension 2 gives rank 4.

Answer 8. Well-definedness and pairing: HRW23, §4.2 and Proposition 4.13, using Lemma 4.12 and Theorem 2.2. Surjectivity: Theorem 1.5 and unobstructedness of the reduced normal form. Word injectivity: Lemma 4.14 using weakly valid open probes. Monodromy injectivity: Corollary 4.10 and Proposition 4.15. Finger moves give suitable bounded, admissible presentations; Proposition 4.16 follows only after classification.

13. Oral examination with model answers

QUESTION 1. State Theorem 4.11 more precisely than "modules are curves." **Answer.** Homotopy classes of *extendable* type D structures correspond bijectively to equivalence classes of *unobstructed* immersed decorated multicurves. The correspondence respects the box-tensor and immersed Floer pairings with the specified reflection convention. It is not a realization theorem for actual manifolds.

QUESTION 2. Why does the classification need nonclosed test objects? **Answer.** A nonclosed test track dual to a word I gives a Floer pairing whose homology dimension determines the occurrence count of I (after correction by shorter words). Thus it detects the full free homotopy word, whereas homology slopes and nonparallel straight-circle pairings cannot.

QUESTION 3. What does "weakly valid" buy us? **Answer.** The dual word test generally defines a valid algebraic pairing without being itself extendable. Theorem 2.2 and the reduced/weakly valid test formalism allow such a probe; demanding that every test arise as a manifold invariant would defeat the injectivity argument.

QUESTION 4. Why do rational slopes not detect monodromy? **Answer.** Incommensurable Floer pairings have dimension $kk'i(\gamma,\gamma')$, independent of A, B. To distinguish monodromy, one uses parallel tests where the differential involves $I + A \otimes B$.

QUESTION 5. Explain Proposition 4.15 in one minute. **Answer.** Regard an invertible matrix as a finite $\mathbb{F}[t,t^{-1}]$-module. Pair against a dual inverse companion matrix for p^ℓ. The kernel of $I + A \otimes B$ is the intertwiner space from $R/(p^\ell)$ to V_A, whose dimension is $\deg p \sum_j \min(\ell, \lambda_{p,j})$. First and second discrete differences recover every elementary-divisor block multiplicity, hence similarity.

QUESTION 6. Why does the double traverse not equal two trivial strands? **Answer.** Primitive reduction produces a cyclic permutation monodromy S, not the identity. In characteristic two its minimal polynomial is $(t+1)^2$, and a parallel trivial-rank-one test detects it by its one-dimensional fixed space.

QUESTION 7. Where does admissibility intervene? **Answer.** Commensurable immersed curves can bound periodic annular domains. A naive minimal drawing may fail the positive-and-negative multiplicity condition, so HRW use admissible positioning or finger moves before defining and comparing bigon complexes. The parallel local-system complex then yields the kernel formula.

QUESTION 8. Which conclusions require the 2022 companion rather than only Theorem 4.11? **Answer.** The graded pairing package (Theorem 5 and phantom arrows), elliptic invariance for manifold invariants (Theorem 7), and exclusion of peripheral components (Proposition 10). These are extra manifold or grading properties, not consequences of abstract ungraded classification alone.

14. Exact source references and audit

[HRW23, main] J. Hanselman, J. Rasmussen, L. Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, arXiv:1604.03466v3, dated 25 April 2023; published as *J. Amer. Math. Soc.* **37** (2024), 391–498, DOI 10.1090/jams/1029. Verified in the arXiv PDF: Definitions 4.1–4.2 at printed pp. 41–42; Definition 4.7 and Lemma 4.8 at p. 44; Theorem 4.9 and Corollary 4.10 at pp. 44–46; Theorem 4.11 and the four equivalence moves at p. 46; Lemma 4.12 and Proposition 4.13 at pp. 47–48; test word construction and Lemma 4.14 at pp. 49–51, Figures 40–41; Proposition 4.15 and Proposition 4.16 at p. 52. **All page numbers here are the printed arXiv-v3 page labels, not journal pagination.**

[HRW22, companion] J. Hanselman, J. Rasmussen, L. Watson, *Heegaard Floer homology for manifolds with torus boundary: properties and examples*, *Proc. London Math. Soc.* **125** (2022), 879–967, DOI 10.1112/plms.12473. Verified against the open-access journal HTML: §2.1, Theorem 5 and Definition 29 (grading arrows/phantom arrows); §3.2, Theorem 7 and Proposition 10 (elliptic involution and peripheral obstruction). The companion is used here to distinguish grading- and manifold-level restrictions from algebraic classification.

[LOT18, supporting only] R. Lipshitz, P. Ozsváth, D. Thurston, *Bordered Heegaard Floer Homology*, *Memoirs Amer. Math. Soc.* **254** (2018), no. 1216, DOI 10.1090/memo/1216. The box-tensor formalism is supporting background only; the two HRW papers remain the primary reading.

Source/derived audit. Exact numbered HRW statements and the admitted figure-dependent portions are explicitly labelled above. The PID proof in §6, the tensor ranks in §7, and the elementary examples are **derived calculations**, not theorems quoted from HRW. The two abstract Jordan-block examples are not claimed to come from geometric three-manifolds.