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Day 19 — Gradings, Spin-c Lifts, and Maslov Rigidity

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Coefficients: $\mathbb{F} = \mathbb{F}_2$.
Files: PDF and source LaTeX are distributed with this lecture.

ABSTRACT Aim and status Day 18 classified *ungraded* extendable type D structures using decorated immersed curves. Today we add the relative Spin^c, Maslov and $\mathbb{Z}/2$ gradings for **actual torus-boundary three-manifolds**. The principal source result is HRW22, Theorem 5; the main-paper input is HRW23, §6, especially Propositions 6.5 and 6.7. Fully proved derived lemmas are distinguished below from source-dependent proof sketches.

1. Sources and the theorem being reconstructed

The two central papers perform complementary tasks. **[HRW23]** establishes the immersed-curve model for $\widehat{CFD}$, geometrizes the $\mathbb{Z}/2$ grading using orientations, and relates the Spin^c splitting to lifted curve positions. **[HRW22]** constructs the *full relative Maslov-grading package* and proves that the resulting grading on immersed-curve Floer homology agrees with the closed-manifold Heegaard Floer grading. The LOT grading group is an algebraic supporting calculation, not our primary object.

Let M be a compact, connected, oriented three-manifold with one torus boundary, with a distinguished puncture $z \in \partial M$; put $T_M = \partial M \setminus \{z\}$. The invariant $\mathcal{C}(M) = \widehat{HF}(M)$ is a finite collection of unobstructed immersed curves with local systems, decomposed into Spin$^c(M)$ summands. The underlying curves are defined only up to the permitted equivalence and their boundary parametrization is auxiliary.

THEOREM Source theorem — HRW22, Theorem 5 For $Y = M_0 \cup_h M_1$, with M_i compact oriented manifolds with torus boundary and h orientation reversing, the immersed-curve pairing identifying $HF(\mathcal{C}(M_0), h(\mathcal{C}(M_1)))$ with $\widehat{HF}(Y)$ is an **isomorphism of relatively graded vector spaces**. The curve pairing has a Spin$^c(Y)$ direct-sum decomposition, and a relative Maslov grading **within each Spin-c summand**, agreeing with the corresponding structures on $\widehat{HF}(Y)$.

THEOREM Source theorem — HRW23, Proposition 6.7 If $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ restricts to $(\mathfrak{s}_0, \mathfrak{s}_1)$, then the $\mathfrak{s}$ summand is calculated by pairing appropriate $\overline{T}_Y$-projections of $\mathcal{C}(M_i, \mathfrak{s}_i)$ after a deck translation depending on $\mathfrak{s}$. The relative $\mathbb{Z}/2$ grading is read from the **signs of intersections** of suitably oriented curves.

The grading theorem is not a claim of a preferred absolute integer Maslov degree for every generator of an arbitrary closed manifold; indeed, a general Y may only carry a relative grading modulo periodic-domain ambiguity. A regular absolute grading on torsion Spin^c summands is additional structure and requires normalization, not merely drawing curves.

2. Covers and relative Spin-c classes

Fix a parametrization (α, β) and let

$$j_* : H_1(\partial M; \mathbb{Z}) \longrightarrow H_1(M; \mathbb{Z}), \qquad H_M = \mathrm{im}(j_*).$$

Let λ be a generator of $\ker j_*$, the *homological longitude*. It need not be primitive. The *rational longitude* is the primitive class spanning its rational slope. Use the standard punctured-plane cover $\widetilde{T} = \mathbb{R}^2 \setminus \mathbb{Z}^2$ and quotient by the translations generated by λ:

$$\overline{T}_M = \widetilde{T}/\langle\lambda\rangle, \qquad H_M \cong H_1(\partial M; \mathbb{Z})/\langle\lambda\rangle.$$

This describes the **lattice cover**; when λ is not primitive, one must retain the torsion in its deck group rather than silently replacing λ by its primitive rational slope. The correct relative grading labels live in an affine set modeled on H_M (or the half-lattice variant below), not in an arbitrarily normalized $\mathbb{Z}^2$.

For each absolute $\mathfrak{s} \in \mathrm{Spin}^c(M)$, the refined relative grading of a bordered generator x depends on its idempotent $\iota(x)$. With a chosen reference generator, the fiber over $\mathfrak{s}$ is modeled on H_M. The two idempotents contribute staggered half-lattice cosets. Informally their union can be read in $\frac{1}{2}H_M$. In the planar model, a switch lifts to a half-lattice point.

WARNING Intrinsic versus relative position The position of **one** connected lifted curve is determined only up to deck translation. To compare generators on *different components in the same absolute Spin-c summand*, one needs extra relative lift information. A chosen origin is a gauge choice, not an invariant.

Derived lemma: comparing paths modulo longitudes

Let Γ_0, Γ_1 be curve collections such that every component of Γ_i has homology a multiple of a fixed class $\lambda_i \in H_1(T^2)$. Suppose $x, y \in \Gamma_0 \cap \Gamma_1$, and choose paths $p_0 : x \to y$ in the first decorated collection and $p_1 : y \to x$ in the second. If one path is changed by traversing an entire closed component, its class changes by a multiple of λ_i. Thus

$$[p_0 + p_1] \ \in \ H_1(T^2; \mathbb{Z})/\langle\lambda_0, \lambda_1\rangle$$

is independent of these particular path changes. **Proof:** the difference of two routes on the same connected closed component is an integral cycle, so its homology is an integral multiple of the component class. With phantom arrows one must keep their prescribed relative positions fixed. This is the topological mechanism underlying the relative Spin-c difference in HRW22, §2.1. □

When $\lambda_0 = (1, 0)$ and $\lambda_1 = (2, 5)$, the quotient has order five because

$$\det \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix} = 5, \qquad H_1(T^2)/\langle \lambda_0, \lambda_1 \rangle \cong \mathbb{Z}/5.$$

This order-five statement is a **homological toy calculation** unless one also supplies manifold modules and the relevant Spin-c restrictions. It does not, by itself, compute Floer differentials.

3. Phantom arrows versus crossover arrows

> **THEOREM** Source definition — HRW22, published Definition 29 For a collection of n immersed components, a *set of grading arrows* consists of $n-1$ extra arrows connecting the components to make their union path-connected. They are called **phantom arrows**. They may be traversed to compare grading labels, but they are **ignored when counting Floer bigons**. In arXiv:1810.10355v1 the same definition is numbered 28.

For example, for three components $\gamma_1, \gamma_2, \gamma_3$, choose two phantom arrows, connecting γ_1 to γ_2 and γ_2 to γ_3. This is a spanning tree on the set of components. The induced paths compare their relative Spin-c lifts and Maslov degrees. If the geometric relative shifts along the chosen arrows are d_{12} and d_{23}, the resulting shift from component 1 to component 3 is $d_{12} + d_{23}$ (with the group-law correction for full bordered gradings; for pure Spin-c translation it is an ordinary sum).

A phantom arrow is **not** an actual crossover arrow retained to encode local-system monodromy: the latter changes which parallel strands are connected by directed paths and hence changes the type D data. Confusing the two changes the differential incorrectly. This issue is acute for the figure-eight complement, whose two components belong to one absolute Spin-c summand (HRW22, Figure 19).

Derived spanning-tree lemma

For n nonempty components, at least $n-1$ graph edges are needed to connect them. Conversely, a tree with $n-1$ edges connects all components and gives a unique graph-theoretic route between every pair. **Proof:** every connected graph contains a spanning tree, whose edge count is $n-1$ (remove one leaf inductively). □

This explains the number of phantom arrows; it **does not** prove that an arbitrary choice of geometric arrow endpoints realizes the same Floer grading. Their homotopy classes and winding data must be compatible with the bordered grading, and changes of choices are controlled by the allowed equivalences.

4. The geometric Maslov formula

Let Γ_0, Γ_1 be two collections in the punctured torus equipped with local systems, relative Spin-c lifts, orientations, and phantom arrows as needed. Let x, y be intersection generators **in the same Spin-c summand**. Choose compatible lifts $\tilde{x}, \tilde{y} \in \mathbb{R}^2 \setminus \mathbb{Z}^2$. Let p_0 be a path from $\tilde{x}$ to $\tilde{y}$ in the lift of Γ_0 and p_1 a path from $\tilde{x}$ to $\tilde{y}$ in the lift of Γ_1, including grading arrows where required. Their oriented difference gives a closed piecewise-smooth cycle

$$C = p_0 - p_1.$$

Use a generic perturbation so the two curve germs at the endpoints meet orthogonally. Define the **signed lattice-point winding**

$$N(C) = \sum_{q \in \mathbb{Z}^2} \mathrm{wind}(C, q),$$

where only finitely many terms are nonzero (the path is compact and avoids the pegs). Let $R(C)$ be the total **rightward/clockwise** tangent rotation on the smooth pieces of C, **omitting the two corner jumps at** x, y.

> **THEOREM** Source construction — HRW22, §2.1 Within a fixed relative Spin-c summand, the geometric relative Maslov grading is
>
> $$\boxed{m(y) - m(x) = 2N(C) + \frac{R(C)}{\pi}.}$$
>
> The formula uses compatible lifts and a fixed grading-arrow decoration. The two corner jumps are excluded from $R(C)$. The equality is relative and subject to the same periodic-domain indeterminacy as the bordered grading.

Sign audit. A positively oriented empty embedded bigon, with its smooth sides meeting orthogonally at endpoints in the HRW convention, has $R(C) = -\pi$ and $N(C) = 0$, so $m(y) - m(x) = -1$. One enclosed puncture adds $+2$ to this difference, giving $+1$. The punctured bigon **does not count** in the hat differential; it still provides a legitimate grading comparison.

Derived computation: several punctures

For the same boundary-turning type, if the enclosed pegs have signed total winding k, then

$$\boxed{\Delta m = 2k - 1.}$$

The cases $k = -1, 0, 1, 2$ give $-3, -1, 1, 3$. This is a calculation of the grading expression, **not** a claim that all these bigons are admissible or holomorphic. In particular, only puncture-free bigons contribute to the usual hat differential.

5. Proof checkpoints for the formula

5.1 Additivity under concatenation — a complete derived lemma

Suppose C_1, C_2 are compact oriented cycles whose concatenation $C_1 * C_2$ is defined, and tangent corner conventions are compatible. Then

$$N(C_1 * C_2) = N(C_1) + N(C_2), \qquad R(C_1 * C_2) = R(C_1) + R(C_2)$$

provided no new uncounted corner is introduced. **Proof:** winding number is additive for sums of 1-cycles by the argument principle/topological degree, pointwise at every peg; summing gives the first identity. A continuous oriented choice of tangent angle along each smooth portion has net angle increment additive under concatenation; the declared corner omissions stay fixed. □

Consequently, Δm behaves like a difference of relative gradings along composable paths. It is **not** generally an absolute grading: a closed periodic route may have nonzero Maslov shift in the permitted grading subgroup. An attempt to prove that the expression is invariant under *all* changes of path, with no periodic-domain quotient, would be incorrect.

5.2 Homotopy through a puncture-free region

Suppose C_t is a generic deformation through cycles avoiding $\mathbb{Z}^2$, keeping the tangent and corner conventions. The integers $\mathrm{wind}(C_t, q)$ are constant as long as the deformation avoids q, while tangent rotation varies continuously but is integer-constrained after fixing the corner angles and closed rotation index. Thus $2N + R/\pi$ is constant. For immersed regular homotopies, care is

required at tangencies and Reidemeister moves: a correct statement is for the **graded immersed-curve object** with its transported grading arrows, not arbitrary ungraded homotopy.

5.3 Source proof of the grading theorem — incomplete sketch

HRW22, §§2.2–2.3 compares the above geometric expression with the noncommutative LOT grading on generators of $\widehat{CFD}$ and $\widehat{CFA}$. For an individual type D coefficient arrow, the Maslov change splits into an area contribution (twice the signed area of a triangle in the lift) and a path contribution $\pm\frac{1}{2}$. Summing along a path yields twice the winding-weighted area plus a signed edge count; passing to immersed curves and including tangent rotation recovers $2N + R/\pi$. The argument is then applied to each box-tensor generator. **Not reproduced:** every coefficient-label case in the source's Table 2, the precise periodic-domain quotient, and the type A/ D identification of idempotent gradings. Those steps are the substantive source proof of HRW22, Theorem 5, not consequences of the elementary additivity lemma alone.

5.4 Exact source arithmetic checks

HRW22, Figure 21 gives a path with area contribution $9/2$ and path contribution -2, yielding

$$\boxed{\Delta m = \frac{9}{2} - 2 = \frac{5}{2}.}$$

Figure 22 has signed area contribution 20 and path contribution -1 (a region of multiplicity two is counted twice), giving

$$\boxed{\Delta m = 20 - 1 = 19.}$$

These are *train-track* grading examples; the half-integral intermediate value is appropriate to the refined bordered Maslov component before the closed pairing normalization. Neither example asserts that a closed-manifold Spin-c summand has half-integral relative Maslov grading.

6. The noncommutative bordered grading group

The bordered grading group G (HRW22, §2.2) consists of triples $(m; i, j)$ satisfying the parity and integrality conditions in the source, with $m, i, j \in \frac{1}{2}\mathbb{Z}$. The pair (i, j) is the Spin-c component, while m is the Maslov component. Its group law includes the **oriented determinant cocycle**:

$$(m; v)(m'; v') = (m + m' + \det(v, v');\ v + v'),$$

with the determinant sign tied to the source's chosen boundary basis. The central Maslov element is $\Lambda = (1; 0, 0)$. In particular, reversing the order of two factors changes their Maslov component by twice their determinant. This noncommutativity is the algebraic form of signed area. A mistake in the orientation of (α, β) reverses the determinant term.

For a type D arrow with coefficient ρ_I, the grading relation is, in the source convention,

$$\mathrm{gr}(\partial x) = \Lambda^{-1}\,\mathrm{gr}(x), \qquad \mathrm{gr}(\rho_I \otimes y) = \mathrm{gr}(\rho_I)\,\mathrm{gr}(y).$$

The grading is defined only up to an overall shift and the action of a periodic-domain subgroup $P(x_0) \subset G$. The box tensor product of a type A and a type D structure uses a double coset $P(x_0)\backslash G / P(x_1)$; the Spin-c part determines the closed summand, and only after matching that

part do the Maslov coordinates give a relative difference. It is not legitimate to subtract arbitrary m coordinates before adjusting the Spin-c components.

Derived determinant calculation

Let $v = (2, 1)$ and $v' = (1, 3)$ in an oriented integral lattice. Then $\det(v, v') = 2 \cdot 3 - 1 \cdot 1 = 5$. In the ambient central-extension formula the products in opposite orders have Maslov terms differing by 10. This is a **cocycle calculation**; the chosen full triples must additionally satisfy the source's half-integrality conditions to define elements of the actual bordered grading group.

7. Mod-two grading and oriented intersections

HRW23, §6.1 starts with the relative $\mathbb{Z}/2$ grading on type D generators, expressed by signs at train-track switches. In a valence-two curve graph this is equivalent to orienting the underlying curves; local-system strands in a connected packet inherit compatible orientation. For two transversely intersecting oriented curves, the local intersection sign is the sign of the ordered pair of tangent vectors, with a fixed convention for the ambient surface orientation.

> **THEOREM** Source theorem — HRW23, §6.1 and Proposition 6.7 After making the type A/type D orientation adjustment, the **relative mod-two grading** of intersection generators is read from their oriented intersection signs. Reversing the orientation of one whole curve collection exchanges all parity labels, i.e. changes the *global* parity normalization, not the relative parity differences.

Proof of a basic parity fact

Let x, y be transverse intersection generators with local signs $\epsilon(x), \epsilon(y) \in \{+1, -1\}$. Define a parity label $\pi(x) \in \mathbb{Z}/2$ by $\epsilon(x) = (-1)^{\pi(x)}$ after fixing a global convention. Then

$$\boxed{\pi(y) - \pi(x) = 0 \text{ if signs agree, and } 1 \text{ if they differ.}}$$

This follows from $(-1)^{\pi(y)-\pi(x)} = \epsilon(y)\epsilon(x)$. The lemma is elementary; identifying these signs with **Floer** parities uses the source's orientation theorem. In particular, it must not be applied to two arbitrarily oriented abstract curve collections without fixing the type A/type D conventions.

An important cross-check is that a puncture-free bigon gives $\Delta m = -1$, an odd difference, and its two endpoints have opposite mod-two parity. A punctured bigon yields $\Delta m = +1$, also odd, although it contributes no hat differential.

8. Three worked manifold computations

8.1 Splice of right-handed trefoil exteriors: rank five and relative Maslov pattern

Use **the specific longitude/meridian gluing described in HRW22, introduction, Figure 1**, not the unconstrained meridian–longitude exchange from the different splicing examples in HRW23, Figure 3. The corresponding two curves have exactly five intersections $x_1, \dots, x_5$, all in one Spin-c summand. HRW22, Example 29 and Figure 17 give puncture-once bigons from x_2 to x_1, x_3 and from x_4 to x_3, x_5. They yield

$$m(x_1) = m(x_3) = m(x_5) = m_0, \qquad m(x_2) = m(x_4) = m_0 - 1.$$

After choosing the relative origin $m_0 = 0$, the pattern is

$$\boxed{(m(x_1), \dots, m(x_5)) = (0, -1, 0, -1, 0).}$$

No puncture-once bigon is counted in the **hat** differential; for the tight minimal-position diagram there is no remaining differential, so the homology rank is five. If the global even/odd normalization makes the three degree-zero generators even, the Euler characteristic is $3 - 2 = 1$, agreeing with the absolute Euler characteristic of an integer homology sphere. This is a consistency check, not an independent proof of the underlying gluing identification.

8.2 Positive 3-surgery on the right-handed trefoil

HRW22, Example 30 and Figure 18 computes $\widehat{HF}(S^3_3(T(2,3)))$ by intersecting the trefoil curve with a slope-three line. There are three intersections, hence rank three. However, to realize all three intersections in the relevant grading cover requires **three different lifts** of the slope-three line. They therefore lie in **different Spin-c summands**. Since $H_1(S^3_3(K)) \cong \mathbb{Z}/3$, this is exactly one generator per Spin-c structure, as required for an L-space surgery.

> **WARNING** Do not fabricate cross-summand Maslov differences The fact that this surgery has rank three does **not** license writing a Maslov difference between arbitrary pairs of its three generators using the relative formula. They lie in different Spin-c structures; absolute d-invariants require additional input.

8.3 Positive 1-surgery on the figure-eight knot

HRW22, Example 31 and Figure 19 depicts a **phantom arrow** connecting the two components of the figure-eight complement invariant. Pairing with the slope-one test circle gives three intersections a, b, c in the same Spin-c summand. The source calculates

$$\boxed{m(b) - m(a) = 1, \qquad m(c) - m(a) = 1.}$$

For $b - a$, a puncture-once bigon has $N = 1$ and $R = -\pi$, so $\Delta m = 2 - 1 = 1$. For the other comparison, the path oriented from c to a in the source has one peg with winding -1 and smooth rightward rotation $+\pi$. Hence $m(a) - m(c) = 2(-1) + 1 = -1$, or equivalently $m(c) - m(a) = +1$. The **verified source conclusion** is $m(b) = m(c) = m(a) + 1$; changing the direction of the path reverses the displayed difference.

The three generators thus have relative pattern $(0, 1, 1)$. An integer homology-sphere Floer group of rank three has mod-two Euler characteristic ± 1 (here one parity contributes once and the other twice), consistent with the formula. No claim of an absolute value of $d(S^3_1(4_1))$ is being made.

9. The graded pairing proof architecture

Input 1 (main paper). HRW23, §§2 and 4 identifies the underlying immersed-curve bigon complex with the bordered type A/type D pairing, while §6.1 gives a geometric orientation/sign convention. In §6.2, Proposition 6.5 identifies the refined Spin-c grading with the positions of lifted train-track switches. Proposition 6.7 reads closed-manifold Spin-c summands from compatible deck translates and their $\mathbb{Z}/2$ gradings from signs.

Input 2 (companion paper). HRW22, §2.2 recalls the noncommutative graded group and its periodic-domain quotient. Section 2.3 compares an individual chord-labelled differential with a planar area-plus-path contribution; it then adds along train-track paths. Phantom arrows keep this comparison possible when the simplified curves are disconnected.

Matching step. Under the generator correspondence of the ungraded pairing theorem, lifted intersection positions match the Spin-c components of tensor generators; the Maslov cocycle becomes twice signed area and signed edge-traversal, which equals $2N(C) + R(C)/\pi$ after the chosen corner convention. Thus the geometric map is grading preserving. This proves HRW22, Theorem 5 **relative to the cited source's algebraic local-arrow checks and periodic-domain analysis**.

PROOF What is proved here and what is cited The quotient-of-paths lemma, spanning-tree lemma, additivity lemma, determinant calculations, and local grading computations are complete. The full comparison with all type D chord coefficients, the quotient by periodic domains, and the precise type A/idempotent signs are a **source-dependent proof sketch**. Theorem 5 itself is a proved theorem of HRW22, not a result newly established in these notes from the elementary lemmas alone.

10. Convention and hypothesis traps

1. **Different Spin-c summands:** relative Maslov differences are defined *within* a fixed summand. Do not subtract degrees across summands without extra normalization.
2. **Primitive versus homological longitude:** $\ker j_*$ may be generated by $n\lambda_{\text{rat}}$ with $n > 1$. Forgetting n loses grading torsion.
3. **Phantom versus real arrow:** phantom arrows transport relative grading data and are omitted from the Floer differential; real crossover arrows can encode monodromy or basis changes.
4. **Corner convention:** $R(C)$ excludes the endpoint right-angle jumps. Counting them changes the grading by one.
5. **Winding signs:** the enclosed pegs are counted with signed integer multiplicity, not unsigned area.
6. **Type A versus type D:** reflection and idempotent conventions can reverse an orientation sign; determine them before interpreting parity.
7. **Homology versus free homotopy:** the spin grading uses an abelianized path difference, but the underlying curve classification from Day 18 detects full free homotopy.
8. **Absolute d-invariants:** Theorem 5 is relatively graded. It does not alone compute absolute correction terms.

11. Seven qualifying-exam exercises

EXERCISE E1 — A quotient of slope classes Suppose $\lambda_0 = (1, 0)$ and $\lambda_1 = (2, 5)$ in $H_1(T^2) \cong \mathbb{Z}^2$. Compute $H_1(T^2)/\langle\lambda_0, \lambda_1\rangle$ and explain how its five elements label *relative* Spin-c differences in a compatible model. **Hint:** Smith normal form or the determinant.

EXERCISE E2 — Phantom-arrow economy An immersed-curve collection has five connected components in one absolute Spin-c summand. Prove the minimum number of grading arrows required to make the decoration path-connected is four. Explain why their homotopy classes, not only their number, matter. **Hint:** spanning trees; change of lifts.

EXERCISE E3 — Bigon Maslov differences For a right-angle bigon with $R = -\pi$, compute Δm for signed peg winding $k = -1, 0, 1, 2$. State which case is eligible for the hat differential and why. **Hint:** $2k - 1$.

EXERCISE E4 — The figure-eight comparison In the figure-eight $+1$-surgery calculation of HRW22, Example 31, explain why the phantom arrow is needed to compare a and c. State the verified relative pattern of their gradings and why all three intersection points lie in one Spin-c summand. **Hint:** Figure 19 and its grading cover.

EXERCISE E5 — Additivity and periodic domains Prove additivity of N under a sum of cycles avoiding lattice points. Explain why this does *not* prove a globally absolute Maslov grading. Describe the correct periodic-domain ambiguity. **Hint:** winding number as a degree and the subgroup $P(x_0) \subset G$.

EXERCISE E6 — The graded torus algebra Compute $\det((2, 1), (1, 3))$, and the difference of Maslov cocycles for multiplication in opposite orders. Explain which part of your calculation is independent of the detailed half-integrality restrictions on G. **Hint:** the antisymmetry of determinant.

EXERCISE E7 — Three surgery Spin-c audit Explain why positive 3-surgery on the right-handed trefoil has three generators in three different Spin-c structures, while the rank-five trefoil splice example has all five generators in a single Spin-c structure. Can one use the same relative Maslov comparison between arbitrary generators in both examples? **Hint:** count deck lifts.

12. Solutions and proof checkpoints

E1 answer

The determinant is five, and $(1, 0)$ is primitive, so the Smith normal form has invariant factors $(1, 5)$. Therefore $\mathbb{Z}^2/\langle(1, 0), (2, 5)\rangle \cong \mathbb{Z}/5$. The five classes are represented by $(0, 0), \dots, (0, 4)$. In manifold pairings these classes organize the *differences* in a fiber of the restriction map on Spin-c structures; the actual affine origin is not canonical.

E2 answer

A connected graph on five vertices has at least four edges, and a tree with four edges exists. One may choose a chain $1 - 2 - 3 - 4 - 5$. Each arrow fixes a relative path/translation between two curve components. Replacing an arrow by another with extra winding around the puncture can change the encoded relative grading even though the connectivity graph is unchanged.

E3 answer

$\Delta m = 2k - 1$ gives $-3, -1, 1, 3$ for $k = -1, 0, 1, 2$. An empty positive bigon ($k = 0$ and no punctures inside) is eligible for the hat differential if it satisfies the usual admissibility and orientation conditions. Puncture-enclosing bigons do not contribute. A negative signed winding is an algebraic orientation convention, not a positive holomorphic disk.

E4 answer

The figure-eight invariant has two components in the same absolute Spin-c summand. A phantom arrow fixes their relative lift position and Maslov shift. HRW22, Example 31 proves $m(b) - m(a) = m(c) - m(a) = 1$; Figure 19 shows all three intersections in the same grading cover. Removing the phantom information while preserving only ungraded components would lose the comparison between c and a.

E5 answer

For any lattice point q not on a cycle, winding number is the degree of the normalized map $C \to S^1$ about q; degrees add under concatenation. Summing over lattice points gives additivity of N. But two routes differing by a periodic domain may have a nontrivial allowable shift in G. The relative grading is defined in a quotient by $P(x_0)$ and, for a gluing, by a double coset.

E6 answer

$\det((2, 1), (1, 3)) = 6 - 1 = 5$. Reversing factors changes this contribution to -5, hence the difference is 10 in the Maslov coordinate of the ambient central extension. The determinant arithmetic is independent of half-integral admissibility conditions; membership of the proposed full triples in G must be checked separately.

E7 answer

The rank-three surgery computation uses three different deck translates of the slope-three circle; the intersections occupy three Spin-c summands. The rank-five splice computation has compatible

lifts containing all five intersections simultaneously, so all belong to one Spin-c summand. The relative Maslov formula compares arbitrary pairs only in the latter example; in the former each summand has a single generator and no nontrivial within-summand comparison.

Proof checkpoints

- □ State HRW22, Theorem 5 without claiming an absolute integer grading.
- □ Explain HRW23, Propositions 6.5 and 6.7 and their role in identifying Spin-c labels.
- □ Derive the Spin-c path-difference quotient and calculate an order-five example.
- □ Distinguish phantom arrows, monodromy crossover arrows, and algebraic type D edges.
- □ Define $N(C)$ and $R(C)$ including orientation and omitted-corner conventions.
- □ Calculate puncture-free and puncture-once bigon grading differences.
- □ Explain the five-intersection trefoil splice pattern, the 3-surgery splitting, and the figure-eight +1 pattern.
- □ State the periodic-domain ambiguity and the limit of the source proof reconstruction.

13. Oral examination with model answers

QUESTION Q1. What is the precise enhancement of the ungraded curve invariant? **Answer.** For each absolute Spin-c summand one records a compatible position of its lifts, a relative $\mathbb{Z}/2$ orientation, and if multiple components occur, phantom arrows transporting their relative Maslov and Spin-c shifts. The grading enhancement recovers the relative bordered grading package, not a canonical absolute Maslov origin.

QUESTION Q2. What is a phantom arrow? Is it counted in the Floer differential? **Answer.** It is a grading-only crossover arrow connecting different immersed components to make the decorated union path-connected. It is used for grading comparison and explicitly **ignored** in bigon counts. A true crossover arrow can instead affect local-system monodromy.

QUESTION Q3. State the Maslov formula and every convention. **Answer.** For x, y in the same Spin-c summand, take compatible lifted paths and form $C = p_0 - p_1$. Then $m(y) - m(x) = 2\sum_q \mathrm{wind}(C, q) + R(C)/\pi$, with clockwise rotation along smooth portions and the two right-angle endpoint corner jumps omitted. The result is relative modulo periodic-domain ambiguity.

QUESTION Q4. Why is an empty bigon degree minus one? **Answer.** Its signed peg winding is zero and its clockwise smooth turning in HRW's convention is $-\pi$, hence $\Delta m = -1$. This is the grading drop expected of the hat Floer differential. A bigon enclosing one peg has $\Delta m = +1$ and is excluded from that differential.

QUESTION Q5. How does the main 2023 paper enter a grading theorem proved in the 2022 companion? **Answer.** HRW23, §6 geometrizes the mod-two orientation and Spin-c lift positions; Proposition 6.7 identifies Spin-c summands and relative parity of closed pairings. HRW22, §2 adds the full Maslov comparison and checks it against the noncommutative bordered group, yielding Theorem 5.

QUESTION Q6. Why cannot the rank-three trefoil surgery generators be compared by this relative formula? **Answer.** They lie in three different Spin-c summands because their lift realizations require three different translates of the surgery circle. Relative Maslov grading is defined within a summand. Absolute correction terms require an additional normalization.

QUESTION Q7. Give a computation requiring phantom grading arrows. **Answer.** The figure-eight complement has two immersed components in one absolute Spin-c class; for +1 surgery, the three intersection generators a, b, c have $m(b) = m(c) = m(a) + 1$. HRW22, Figure 19 shows the phantom arrow needed to transport the comparison between the components.

QUESTION Q8. What is the main logical gap in a proof using only $2N + R/\pi$? **Answer.** One must establish its agreement with all local type A/type D grading changes, handle periodic-domain indeterminacy, and verify invariance under the allowed graded train-track moves. Additivity of the formula alone does not imply Theorem 5; these checks are in HRW22, §§2.2–2.3.

14. Reading plan and exact references

Four-hour-fifty-minute allocation: 35 min on HRW23, §6.1–6.2; 55 min on HRW22, §2.1 and Figures 17–19; 55 min on HRW22, §2.2, Tables 2–3 and the group G; 50 min on HRW22, §2.3 and Figures 21–22; 45 min on worked computations and exercises E1–E4; 50 min on exercises E5–E7 and oral defense. Total: **290 min**.

[HRW22] Jonathan Hanselman, Jacob Rasmussen, Liam Watson, *Heegaard Floer homology for manifolds with torus boundary: properties and examples*, *Proc. London Math. Soc.* **125** (2022), 879–967. Primary reading: §2.1–2.3, Theorem 5, **published Definition 29**, examples and figure numbers can differ between published and preprint versions. In the directly inspected arXiv:1810.10355v1, the matching passages are **Definition 28, Examples 29–31, Figures 17–19, 21–22**, printed pp. 19–28. The equations and examples in this lecture are cross-checked against that *v1* text.

[HRW23] Jonathan Hanselman, Jacob Rasmussen, Liam Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, March 8, 2023 author manuscript, §6.1–6.3, Proposition 6.5 and Proposition 6.7; see also arXiv:1604.03466. The author manuscript is a directly examined 2023 main-paper version; pagination need not coincide with journal printing.

[LOT18, supporting] Lipshitz, Ozsváth, Thurston, *Bordered Heegaard Floer Homology*, *Memoirs AMS* **254** (2018), no. 1216, Chapters 10–11. Used only for the algebraic grading formalism invoked by the two HRW papers.

Next lecture: Day 20 — a full closed-book synthesis of train-track classification, pairing, Spin-c grading, and L-space interval applications.