HRW Qualifying Examination - Day 20

Closed-Book Proof Reconstruction: Torus Algebra to L-Space Gluing

11 November 2026

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Course date: Wednesday, 11 November 2026. **Study allocation:** 4 hours 50 minutes. **Coefficients:** $\mathbb{F} = \mathbb{F}_2$.
PDF: HRW_Day20.pdf · **Next note:** HRW Day 21 · **Prerequisites:** HRW Day 17, HRW Day 18, HRW Day 19.

1. Objectives and the two-paper theorem map

Concept atlas: Torus Algebra · Extendable Type D Structure · Crossover Arrows · Local Systems · Immersed-Curve Pairing · Spin-c Grading · Turaev Torsion · Thurston Norm · L-space Intervals.

This is a **closed-book reconstruction lecture**, not a replacement for reading the papers. The two primary sources have already been read once. We now aim to state every central theorem with its hypotheses, derive its usable computational consequences, and identify the exact point where a geometric source argument is needed. Work in the smooth oriented category. An *HRW manifold with torus boundary* is compact, oriented, and has one torus boundary component; the gluing applications may require irreducibility or boundary incompressibility in addition. The punctured torus is $T_M = \partial M \setminus \{z\}$.

The **2023 main paper** establishes: the manifold invariant $\widehat{CFD}(M)$ is extendable (Theorem 1.4); extendable type D structures admit immersed-curve representatives with local systems (Theorem 1.5); the equivalence with unobstructed decorated curves is a **classification** (Theorem 4.11); the geometric pairing gives $\widehat{HF}$ for torus gluings (Theorems 1.2 and 2.2, with their distinct scopes); and the tangent-slope and L-space gluing theorems give geometric applications (Theorems 7.17 and 1.14). The **2022 companion** proves the relatively graded refinement (Theorem 5), elliptic invariance for *manifold* invariants (Theorem 7), knot Floer/filling-core interpretations (Theorem 11 and Proposition 12), signed-ray Turaev torsion (Proposition 62), the Thurston-norm obstruction-height formula (Proposition 65), and torsion determination of a Floer-simple L-space interval (Theorem 69).

> **warning.** Logical classification of statements A statement about **all extendable type D structures** is not automatically a statement about all modules realized by manifolds, or conversely. The curve classification does **not** imply arbitrary local systems or peripheral circles are realized by manifolds. The geometric gluing theorem has an elliptic-involution correction in the initial 2023 formulation; HRW22's elliptic symmetry removes it for actual manifold invariants. Distinguish a *closed filling-slope circle* used for $\widehat{HF}(M(\alpha))$ from a *proper noncompact slope arc* used for $\widehat{HFK}$ of the filling core.

Here is the proof-dependency chain that the examiner may ask you to draw:

$\widehat{CFD}(M)$ extendable (HRW23, Thm. 1.4)
$\Downarrow$
train tracks with crossover arrows (Prop. 3.8)
$\Downarrow$
finite-depth arrow reduction (Prop. 3.13)
$\Downarrow$
decorated immersed curves (Thm. 1.5)
$\Downarrow$
pairing-compatible classification (Thm. 4.11)
$\Downarrow$
geometric gluing (Thm. 1.2), graded by HRW22 Thm. 5
$\Downarrow$
torsion, norm and L-space filling; L-space gluing.

The next eight stations should take roughly **three hours of active mathematical reconstruction**. The written and oral examinations supply the remaining time.

2. Station A — the torus algebra, curvature, and extendability

Fix an ordered parametrization of the oriented boundary torus. The torus algebra $\mathcal{A}$ has orthogonal idempotents ι_0, ι_1 and six non-idempotent generators $\rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}$. The nonzero products of non-idempotent generators are exactly

$$\rho_1\rho_2 = \rho_{12}, \qquad \rho_2\rho_3 = \rho_{23}, \qquad \rho_1\rho_{23} = \rho_{123}, \qquad \rho_{12}\rho_3 = \rho_{123}.$$

All remaining such products vanish; multiplication by compatible idempotents is understood. A

finite type D structure is an idempotent-graded $\mathbb{F}$-vector space V and a map $\delta^1 : V \to \mathcal{A} \otimes_{\mathcal{J}} V$ satisfying

$$(\mu \otimes 1)(1 \otimes \delta^1)\delta^1 = 0.$$

Complete calculation (an invalid two-arrow graph). Take idempotent-compatible generators x, y, z and suppose $\delta^1(x) = \rho_1 \otimes y$, $\delta^1(y) = \rho_2 \otimes z$, and no other path from x to z contributes. Then

$$(\mu \otimes 1)(1 \otimes \delta^1)\delta^1(x) = \rho_1\rho_2 \otimes z = \rho_{12} \otimes z \neq 0.$$

Thus the graph is not a valid type D structure. One cannot read a labelled graph as a module without checking all length-two products and cancellations.

The *extended* torus algebra $\widetilde{\mathcal{A}}$ restores a cyclic fourth chord ρ_0; its distinguished central curvature element is

$$U = \rho_{1230} + \rho_{2301} + \rho_{3012} + \rho_{0123}.$$

An extended differential satisfies $\tilde{\partial}^2 = U\,\mathrm{id}$. Its reduction by $\rho_0 = 0$ has square zero. **Extendability** means that the ordinary type D structure admits such a curved lift up to the specified homotopy notion. HRW23, Theorem 1.4 supplies this for manifold $\widehat{CFD}$; it is not true by definition for arbitrary type D modules.

> **proof.** Derived auxiliary matrix factorization (complete) Over $R = \mathbb{F}_2[U]/(U^2)$ put $M = \begin{pmatrix} 0 & 1 \\ U & 0 \end{pmatrix}$. Direct multiplication gives $M^2 = UI_2$. Modulo U, the matrix becomes $\bar{M} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ with $\bar{M}^2 = 0$. This is a finite-dimensional *model of the curvature-to-ordinary reduction*, not a claim that every arbitrary 2×2 matrix factorization is a torus-bordered module.

Proof checkpoint A. Explain why a curved differential and a square-zero differential cannot be identified before the quotient. State both the algebraic condition (HRW23, Definition 3.1) and the manifold existence theorem (Theorem 1.4).

3. Station B — from crossover arrows to local systems

An extended type D module produces a train-track picture with switches and labelled strands. HRW23, Proposition 3.8 reduces an appropriate valid reduced train track to immersed strands (valence two at switches) together with *crossover arrows*. It does not say that the arrows vanish. Sliding arrows changes their apparent locations and can create **composition arrows**. HRW23, Proposition 3.13 raises the finite depth of remaining arrows. In the termination argument, arrows between globally distinguishable strands are eliminated, whereas arrows among parallel strands survive as a local system. The geometric induction on arrow depth is substantial and diagram dependent.

> **proof.** Source proof sketch — HRW23, Propositions 3.8 and 3.13 First perform elementary changes of basis that put the relevant strand matrix into the local normal form; record residual off-diagonal data as crossover arrows. Next order arrows by their forward and backward comparison depths. Slide a minimal-depth arrow across a sequence of switches; whenever arrows cross in the appropriate order, add the composition arrow. The local weight comparison proves that newly created arrows do not have smaller depth. Iterate. The remaining parallel-strand packet carries an invertible monodromy matrix. **Incomplete:** the exact depth definition, all four stages of the arrow-slide induction, and the finite termination argument require the diagrams and lemmas of HRW23, §§3.3–3.7; they are not proved from first principles here.

Choose a transverse cut through a packet of k parallel strands. Their fiber is $V \cong \mathbb{F}^k$. Tracing

the packet once gives an automorphism $A \in GL_k(\mathbb{F})$. A basis change P sends it to $P^{-1}AP$. Hence the **similarity class**, not an entrywise matrix, is intrinsic. Reversing the curve orientation replaces A by A^{-1}.

Complete computation (a nontrivial local system). Let

$$A = I_2, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $\chi_A(t) = (t+1)^2$ while $\chi_B(t) = t^2 + t + 1$ over $\mathbb{F}_2$. Therefore A and B are not similar, even though their ranks and underlying geometric circles coincide. In fact $B^3 = I_2$, whereas A has order one. Torsion or ordinary homology class will generally forget this distinction; parallel Floer tests can recover it.

Proof checkpoint B. State precisely why Theorem 1.5 proves **existence**, while Theorem 4.11 is needed for **uniqueness**. Do not infer realization of every abstract decorated curve by a manifold.

4. Station C — uniqueness and similarity detection

HRW23, Theorem 4.11 is a bijection between homotopy-equivalence classes of extendable type D structures and the appropriate equivalence classes of unobstructed decorated immersed multicurves, compatible with pairings. Its injectivity has two steps: open test tracks recover weighted **cyclic free-group words** (Lemma 4.14), and parallel local-system tests recover **monodromy similarity classes** (Proposition 4.15). Merely knowing a loop's homology slope cannot determine its free homotopy class in the punctured torus, whose fundamental group is free of rank two.

The key parallel computation has an elementary interpretation. Write $R = \mathbb{F}[t, t^{-1}]$. A matrix $A \in GL_k(\mathbb{F})$ makes V_A an R-module via $t \cdot v = Av$. A second monodromy C acts on W_C. The intertwining operator

$$d : \mathrm{Hom}_{\mathbb{F}}(W, V) \to \mathrm{Hom}_{\mathbb{F}}(W, V), \qquad d(X) = AX - XC$$

has kernel $\mathrm{Hom}_R(W_C, V_A)$. The two-term circle cochain complex with this differential has equal-dimensional degree-zero and degree-one cohomology, so

$$\boxed{\dim H^* = 2 \dim \mathrm{Hom}_R(W_C, V_A).}$$

In the parallel embedded-curve Floer model this is the local-system contribution, after orientation/dual conventions. For a self-pairing, the equation becomes $AX = XA$; hence $\dim HF(\gamma_A, \gamma_A) = 2 \dim \mathrm{Cent}(A)$.

> **proof.** Derived primary-block test (complete) Let $p \in \mathbb{F}[t]$ be irreducible with $p(0) \neq 0$ and degree d. Every R-linear map $R/(p^\ell) \to R/(p^a)$ is determined by $x = f(1)$ with $p^\ell x = 0$. If $\ell \geq a$, all da coordinates are allowed. If $\ell < a$, the allowed elements form $(p^{a-\ell})/(p^a)$ of dimension $d\ell$. Thus
>
> $$\dim_{\mathbb{F}} \mathrm{Hom}_R(R/(p^\ell), R/(p^a)) = d \min(\ell, a).$$
>
> For distinct irreducibles the Hom space vanishes by coprimality. If $V_A = \bigoplus_{p,j} R/(p^{\lambda_{p,j}})$, the test dimensions $D_p(\ell; A) = d \sum_j \min(\ell, \lambda_{p,j})$ determine, by first differences, the number of blocks with $\lambda_{p,j} \geq \ell$, and hence all block multiplicities. Use the inverse transpose of the companion matrix of p^ℓ as B to identify $D_p(\ell; A)$ with $\dim \ker(I + A \otimes B)$. Two nonsimilar matrices give different test ranks. This supplies a complete proof of the algebraic separation input of HRW23, Proposition 4.15.

Worked numerical test. Let J_a be the unipotent Jordan block of length a over $\mathbb{F}_2$, and put $A_1 = J_4 \oplus J_2 \oplus J_2$, $A_2 = J_4 \oplus J_3 \oplus J_1$. Both have characteristic polynomial $(t+1)^8$, minimal

polynomial $(t+1)^4$, and three-dimensional fixed space. The $p = t+1$, $\ell = 2$ test has

$$D_p(2; A_1) = 2+2+2 = 6, \qquad D_p(2; A_2) = 2+2+1 = 5.$$

Taking $B = (J_2^{-1})^T$, the two 16×16 operators $I + A_i \otimes B$ have ranks 10 and 11. Thus even characteristic polynomial, minimal polynomial, and the simplest rank-one parallel test do not determine the local system; the full family does.

Proof checkpoint C. Be able to explain both parts of injectivity and why HRW23, Proposition 4.16 (uniqueness of extension up to homotopy) is a **consequence** of classification, not an assumption.

5. Station D — geometric pairing, generators, and bigons

For two appropriate train tracks, HRW23, **Theorem 2.2** assumes the first is reduced and the second is *special bounded*. It identifies the intersection/bigon chain complex with the algebraic type A/type D box tensor product. Boundedness is needed so the algebraic differential has finite sums. This theorem is logically different from the final three-manifold pairing theorem (HRW23, Theorem 1.2), whose proof also uses the curve classification, admissibility, parametrization-change invariance and the elliptic convention.

> **proof.** Source proof sketch — matching a bigon to a box-tensor operation An intersection of the type A and type D geometric realizations corresponds to an idempotent-compatible tensor generator $x \otimes y$. One side of a bigon records a sequence of torus-algebra chords $\rho_{I_1}, \dots, \rho_{I_k}$ in an iterated type D coefficient. The other side records the matching A_∞ multiplication $m_{k+1}(x, \rho_{I_1}, \dots, \rho_{I_k}) = x'$. Their composition yields the differential term $x' \otimes y'$. **Incomplete:** exact label reversals, small unit-labelled bigons and all possible local configurations are checked in HRW23, §2.6, Figures 10–12.

Complete worked determinant calculation. Let $u = (2, 3)$ and $v = (1, 4)$ be primitive slopes. Straight embedded circles of these directions intersect minimally

$$i(u, v) = \left|\det \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\right| = |8 - 3| = 5.$$

For rank-one trivial local systems in straight minimal position there are no immersed bigons, hence $\dim HF = 5$. If they carry local systems of ranks two and three, each point contributes $2 \cdot 3 = 6$ generators. The circles are incommensurable and the Floer differential vanishes, so $\dim HF = 30$. This is an **abstract decorated-curve calculation**; it does not assert that arbitrary chosen local systems arise from two solid tori.

> **warning.** Parallel circles Two parallel rank-one trivial circles have minimal geometric intersection 0 but $\dim HF = 2$ after the admissible parallel-curve calculation. Never assert $\dim HF = i$ without the noncommensurability/minimal-position assumptions. The sign and inverse-transpose conventions for monodromies also matter.

Proof checkpoint D. Distinguish the chain-level theorem (special-bounded hypothesis), the curve classification, and the manifold geometric pairing theorem. State where the elliptic involution appears, and why HRW22, Theorem 7 allows it to be omitted for *manifold* invariants.

6. Station E — the Spin-c and Maslov grading package

The main paper, HRW23, §6 and Propositions 6.5–6.7, identifies refined Spin^c data with positions of lifted switches in the relevant cover and relative mod-two grading with oriented intersection signs. The **2022 companion** makes the full Maslov grading geometric (Theorem 5 and §2). For an absolute Spin^c summand whose curve invariant has n components, grading/phantom arrows connect the components as a spanning tree of $n-1$ arrows (published Definition 29). They record relative grading offsets and **are not counted** as crossover arrows in the Floer bigon differential.

For x, y in the **same** closed-manifold Spin^c summand, choose compatible planar lifts and a closed

path C assembled from paths along the two decorated curve systems, including grading arrows where required. Let $N(C)$ be total signed lattice-puncture winding and $R(C)$ be clockwise tangent rotation along smooth portions, **omitting corner jumps at the two intersections**. Then HRW22, §2.1 gives

$$\boxed{m(y) - m(x) = 2N(C) + \frac{R(C)}{\pi}.}$$

The result is interpreted with the appropriate periodic-domain grading indeterminacy. It is not a canonical absolute integer grading across different Spin^c structures.

Complete local calculation. For a convex right-angle bigon with no puncture, $N = 0$ and $R = -\pi$, so $\Delta m = -1$. If the same turning pattern encloses one positively wound puncture, $N = 1$ and $\Delta m = 1$. The latter bigon is **not** counted by the hat differential because it covers the marked point. For winding k with the same turning pattern one obtains $\Delta m = 2k - 1$, but this is a grading computation, not evidence that every such region is a positive Floer domain.

Additional algebraic convention. The bordered grading group is a central extension whose multiplication contains an oriented determinant cocycle; double-coset/periodic-domain quotients control the comparisons. Consequently, a naive subtraction of grading labels or unrestricted comparison across Spin^c summands is not justified. The 2022 companion supplies this structure rather than replacing the original bordered grading theory by a single absolute number.

Proof checkpoint E. Explain the difference between an actual crossover arrow (possibly local-system data) and a phantom arrow (grading data), and why $n - 1$ arrows suffice graph-theoretically but not every geometric choice gives the same grading.

7. Station F — Turaev torsion and a complete trefoil calculation

Let M be a rational homology solid torus, with compatible homology orientation. The normalized relative Turaev torsion is an integer-valued function on Spin$^c(M, \partial M)$, an affine $H_1(M)$-set. The relative sutured Spin^c torsor for slope μ is a **different boundary-condition torsor**. HRW22, Lemma 61 identifies compatible pairs whose torsion values satisfy a finite-difference relation; they must not be naively identified without fixing the affine correspondence.

> **theorem.** HRW22, Proposition 62 (source) For a relative Spin^c structure $\hat{\mathfrak{s}}$, the Turaev torsion equals the algebraic intersection of the oriented immersed curve in the refined grading cover with the downward ray $\gamma_{\hat{\mathfrak{s}}}$ from the corresponding lifted puncture:
>
> $$\tau_M(\hat{\mathfrak{s}}) = [\gamma_{\hat{\mathfrak{s}}}] \cdot [\mathcal{C}(M, \mathfrak{s})].$$
>
> **proof.** Conditional complete finite-difference proof Define $n_s = [\gamma_s] \cdot [\mathcal{C}]$. The geometric filling-core pairing yields $n_{s+\mu} - n_s = \chi SFH(M, \gamma_\mu, \bar{s})$. The sutured-Floer decategorification theorem and HRW22, Lemma 61 give the identical difference $\tau_M(s+\mu) - \tau_M(s)$. Hence $d_s = n_s - \tau_M(s)$ is constant on each μ-orbit. For sufficiently negative translates of a fixed s, the downward ray misses the compact curve and the normalized torsion vanishes; so $d_s = 0$ on that orbit. Hence $n_s = \tau_M(s)$ throughout. This proves the proposition **conditional on the cited pairing, decategorification, and torsion-normalization inputs**, not independently of them.

Trefoil computation. For the right-handed trefoil $K = T(2, 3)$, fix the standard symmetric polynomial $\Delta_K(t) = t^{-1} - 1 + t$. HRW22's normalized knot-exterior torsion series is

$$T_K(t) = \sum_{j \in \mathbb{Z}+1/2} \tau_j t^j = \frac{t^{1/2}\Delta_K(t)}{1-t} = \frac{t^{-1/2} - t^{1/2} + t^{3/2}}{1-t}.$$

Since $t^{-1/2} - t^{1/2} = t^{-1/2}(1-t)$,

$$\boxed{T_K(t) = t^{-1/2} + \frac{t^{3/2}}{1-t} = t^{-1/2} + t^{3/2} + t^{5/2} + \cdots.}$$

Thus $\tau_j = 0$ for $j \le -3/2$, $\tau_{-1/2} = 1$, $\tau_{1/2} = 0$, and $\tau_j = 1$ for $j \ge 3/2$. The discrete derivative gives

$$a_n = \tau_{n+1/2} - \tau_{n-1/2}, \qquad (a_{-1}, a_0, a_1) = (1, -1, 1),$$

recovering the three Alexander coefficients. This is a concrete instance of torsion being a **discrete antiderivative** of the Alexander polynomial. Torsion is *not* always $\{0,1\}$-valued outside the Floer-simple setting.

Proof checkpoint F. Derive the discrete-derivative formula without consulting the notes. Explain why unsigned geometric intersections are insufficient: a generic tangency creates two opposite-sign intersections, so the algebraic count is stable whereas the unsigned count changes by two.

8. Station G — Thurston norm, L-space slopes, and gluing

8.1. The norm theorem, with an important exception

For a surface Σ set $\chi_-(\Sigma) = \sum_i \max(0, -\chi(\Sigma_i))$ and $x_M(a) = \min_{[\Sigma]=a} \chi_-(\Sigma)$. HRW22, Proposition 65 studies the maximum relative first-Chern-class evaluation among lifted punctures obstructed from reaching the positive end of a grading cylinder. In its prescribed setup, this maximum recovers $-\chi(\Sigma)$ for a norm-minimizing representative; the proof uses the **proper filling-core test arc** and the external knot Floer norm-detection theorem.

> **warning.** A disk is a genuine convention test For one-boundary-component genus-g surfaces, $-\chi = 2g - 1$. At $g = 0$, $-\chi(D^2) = -1$, but $\chi_-(D^2) = 0$. Do **not** silently rewrite a theorem involving $-\chi(\Sigma)$ as an unqualified formula for x_M in boundary-compressible or solid-torus-like cases. The source has separate treatment of those cases.

> **proof.** Source proof sketch — HRW22, Proposition 65 Tighten the lifted curve to a pegboard and identify its highest obstructing corner peg. The proper arc of rational-longitude slope paired with the curve computes $\widehat{HFK}$ of the filling core (HRW22, Proposition 12 and Theorem 52). Above the top corner the relevant pairing vanishes; at the extremal corner the source's local-bigon argument produces a surviving class. Knot Floer norm detection identifies the extremal Chern evaluation with the surface complexity. **Incomplete:** the corner-survival argument and the exceptional no-corner/solid-torus-like cases require HRW22, §5.2 and its diagrams.

8.2. Interior L-space slopes

For $\alpha \in \mathbb{Q}P^1$, write $M(\alpha)$ for Dehn filling and define $\mathcal{L}(M) = \{\alpha : M(\alpha) \text{ is an L-space}\}$. HRW23, **Theorem 7.17**, identifies the **interior** $\mathcal{L}^\circ(M)$ with the complement, in $\mathbb{Q}P^1$, of the rational **singular essential tangent slopes** $S^{\text{sing}}_{\mathbb{Q}}(M)$. It does *not* state that every nonsingular rational slope is a filling endpoint or that $\mathcal{L}(M)$ equals its interior. A nontrivial local system or certain disconnected curve collections force maximal tangent obstructions under the source's definitions.

HRW22, **Lemma 66**: given an interior L-space slope α, the tightened lift of the relevant Floer-simple curve is a graph in α-adapted coordinates. Torsion values 0 and 1 color the lifted punctures black and white. HRW22, **Proposition 67** relates the singular-slope interval to correctly ordered black–white peg directions, and **Lemma 68** identifies their union with the projectivized boundary preimage of the torsion difference set $D^\tau(M)$.

> **theorem.** HRW22, Theorem 69 (source; exact scope) If M is Floer simple and **not solid-torus-like**, and $\alpha \in \mathcal{L}^\circ(M)$ is a *known* interior slope, then $\mathcal{L}^\circ(M)$ is the largest projective slope interval containing α and avoiding the projectivized nonzero elements of $j_*^{-1}(D^\tau(M))$. This gives the interior, not automatic endpoint membership.

The preimage matters. For the algebraic map $j_* : \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}/3$ defined by $j_*(a, b) = (a, [b]_3)$,

$$j_*^{-1}(2, [1]_3) = \{(2, 1 + 3k) : k \in \mathbb{Z}\}.$$

Taking only the lift $(2, 1)$ loses the infinitely many projective directions $[2 : 1 + 3k]$. This is a **derived homology example**, not a claimed torsion difference set of a particular three-manifold.

8.3. The torus-gluing theorem

> **theorem.** HRW23, Theorem 1.14 (source) Let M_0, M_1 have **incompressible torus boundary**, and let $h : \partial M_1 \to \partial M_0$ reverse orientation. Then $Y = M_0 \cup_h M_1$ is an L-space **if and only if**
>
> $$\mathcal{L}^\circ(M_0) \cup h(\mathcal{L}^\circ(M_1)) = \mathbb{Q}P^1.$$
>
> The hypothesis is not merely that the two sets overlap. Floer simplicity is *not* assumed at the outset.
>
> **proof.** Source proof sketch — HRW23, §7.6 If the gluing is not an L-space, two surviving intersection classes in a Spin^c summand can be connected by paths whose difference is null-homologous in the torus. Compatible lifts have common endpoints; the chord/Rolle argument forces a common tangent direction. Compactness of tight pegboard limits and rational extremal directions produce a rational common singular slope, hence an uncovered slope. Conversely an uncovered rational slope is singular on both sides. HRW's three local pegboard cases (two peg wraps, a corner with nonhorizontal branches, and long horizontal branches) force surviving opposite-parity intersections, obstructing minimal Floer rank. **Incomplete:** the full admissibility and survival analysis is the content of Figures 62–63 and cannot be replaced by elementary Rolle's theorem alone.

A useful abstract circle test. Parametrize $\mathbb{R}P^1$ by $\mathbb{R}/\mathbb{Z}$ (as a circle, not as a literal surgery-slope coordinate). The open arcs $(0.1, 0.8)$ and $(0.6, 1.2)$ modulo one cover the whole circle; the arcs $(0.1, 0.6)$ and $(0.5, 0.9)$ overlap but do **not** cover. These are *topological test arcs*, not asserted L-space intervals of specific manifolds.

Convention checkpoint. On the right-handed trefoil exterior, the positive L-space surgery coefficients are $r \geq 1$ together with the meridional slope ∞ (the latter yields S^3). In the projective **slope-circle** topology, ∞ lies at the **endpoint** of the positive arc, not in its interior: approaching ∞ from the negative-coefficient side leaves the L-space set. Do not treat the real line's $+\infty$ as an interior point of the projective circle. This example is a useful endpoint audit.

9. Station H — applications and hypothesis audit

The main paper's **Theorem 7.15** gives a pinch rank inequality $\dim \widehat{HF}(Y) \geq nm \dim \widehat{HF}(Y_0)$ for a rational-homology-sphere torus decomposition into rational homology solid tori, with n the order of the second piece's rational-longitude image and $m = |H_1(M_1, \partial M_1)|$. The integer homology-sphere pinch gives $n = m = 1$ (Theorem 1.12). The result is **not** a theorem about all degree-one maps.

Theorem **1.8** says a *separating essential torus* forces $\dim \widehat{HF}(Y) \geq 5$. The proof depends on manifold-specific restrictions on curve systems and the exceptional annular admissibility configurations. An arbitrary pair of straight essential torus curves may intersect once, so the rank-five conclusion is not purely elementary planar topology. Theorem **1.15** gives the satellite obstruction: if a nontrivial satellite knot $P(C)$ is an L-space knot, its pattern knot $P(U)$ and its companion C are L-space knots. This is a **necessary**, not sufficient, condition.

> **warning.** Six false deductions to reject under examination 1. "Every type D graph is valid" — false unless $\delta^2 = 0$ and idempotents match.
> 2. "Every extendable module is manifold-realizable" — not asserted; HRW22, Proposition 10 excludes peripheral circles for manifolds.
> 3. "Torsion determines all local systems" — false: it decategorifies curve data.

4. "Minimal intersections always equal Floer rank" — false for parallel/commensurable components.
5. "The Thurston norm is always $-\chi$" — false for disks and spheres.
6. "Overlap of L-space intervals suffices for gluing" — false: full coverage of $\mathbb{Q}P^1$ is required.

10. Written examination — eight exercises with hints

exercise. 1. Validity of a labelled type D graph (20 min). Check the two-arrow graph $x \xrightarrow{\rho_1} y \xrightarrow{\rho_2} z$. Give the coefficient of z in $\delta^2(x)$. Then explain how *two* paths contributing the same algebra coefficient could cancel over $\mathbb{F}_2$. **Hint:** use $\rho_1\rho_2 = \rho_{12}$.

exercise. 2. Curvature and change of basis (20 min). In $R = \mathbb{F}_2[U]/(U^2)$ verify $M^2 = UI_2$ for $M = \begin{pmatrix} 0 & 1 \\ U & 0 \end{pmatrix}$. For $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ compute PMP^{-1} and its square. **Hint:** $P = P^{-1}$.

exercise. 3. A local-system test that elementary invariants miss (30 min). Compare $A_1 = J_4 \oplus J_2 \oplus J_2$ and $A_2 = J_4 \oplus J_3 \oplus J_1$. Compute characteristic/minimal polynomials, fixed spaces, and $\operatorname{rank}(I + A_i \otimes (J_2^{-1})^T)$. Explain the connection with HRW23, Proposition 4.15. **Hint:** $\sum_j \min(2, \lambda_j)$.

exercise. 4. Immersed intersections and local systems (20 min). For primitive slopes $(2, 3)$ and $(1, 4)$ calculate the geometric intersection and rank-two/rank-three incommensurable Floer dimension. Contrast with two parallel trivial rank-one curves. **Hint:** determinant and parallel circle cohomology.

exercise. 5. Maslov and Spin^c audit (25 min). Evaluate $2N + R/\pi$ when $R = -\pi$ and $N = 0, 1, -1$. Explain which corresponding bigon can count in $\widehat{CF}$ and why the calculation requires compatible lifts in the *same* Spin^c summand. **Hint:** point constraints and periodic domains.

exercise. 6. Trefoil torsion as an antiderivative (25 min). Starting from $\Delta(t) = t^{-1} - 1 + t$, expand $T(t) = t^{1/2}\Delta(t)/(1 - t)$ at $t = 0$; list all exceptional τ_j values and reconstruct the Alexander coefficients by finite differences. **Hint:** isolate $t^{-1/2}(1 - t)$ first.

exercise. 7. Boundary preimages and slope topology (25 min). For $j_*(a, b) = (a, [b]_3)$ compute $j_*^{-1}(2, [1]_3)$; determine when $(2, 1 + 3k)$ is primitive. Explain why the points $(0.1, 0.8)$ and $(0.6, 1.2)$ modulo one form a covering union of open arcs. **Hint:** check parity of $1 + 3k$.

exercise. 8. One-page proof-dependency audit (30 min). Draw arrows between HRW23 Theorems 1.4, 1.5, 2.2, 4.11, 1.2, 7.17, 1.14 and HRW22 Theorems 5, 7, Proposition 62, Proposition 65, Theorem 69. Give one missing hypothesis and one nontrivial external input for each of the final three applications. **Hint:** distinguish *chain comparison*, *classification*, *manifold pairing*, and *decategorification*.

11. Answers and proof checkpoints

Exercise 1

The only length-two path contributes $\rho_{12} \otimes z \neq 0$, violating the type D equation. If two different paths with the same source and target each contributed $\rho_{12} \otimes z$, their sum would vanish in characteristic two. The idempotent and all other path coefficients must still be checked; cancellation of a single term is not sufficient in general.

Exercise 2

$M^2 = UI_2$. Since $P^{-1} = P$, multiplication gives $PMP^{-1} = \begin{pmatrix} U & 1+U \\ U & U \end{pmatrix}$. Conjugation yields $(PMP^{-1})^2 = P(UI_2)P^{-1} = UI_2$. Reducing $U = 0$ makes both matrices square-zero. These are ring-matrix facts; no manifold realization is inferred.

Exercise 3

Both matrices have $\chi(t) = (t+1)^8$, $m(t) = (t+1)^4$ and three Jordan blocks, hence $\dim\ker(I + A_i) = 3$. The test block J_2 yields tensor nullities 6 and 5 on a 16-dimensional space, so the ranks are $\boxed{10 \text{ and } 11}$. The module Hom calculation recovers all elementary-divisor block sizes.

Exercise 4

$|2 \cdot 4 - 3 \cdot 1| = 5$. With rank-two and rank-three systems, the nonparallel Floer dimension is $5(2)(3) = \boxed{30}$. For parallel trivial rank-one curves, minimal geometric intersection is 0 but the admissible Floer dimension is 2; it is the circle cohomology with trivial coefficients.

Exercise 5

The grading differences are $-1, 1, -3$ for $N = 0, 1, -1$. Only the **puncture-free** bigon, under the usual positivity and admissibility conditions, is eligible for the hat differential. The others are grading calculations and may be punctured or non-positive domains. No relative Maslov difference is defined by this construction between arbitrary generators in different Spin^c summands.

Exercise 6

$$T(t) = t^{-1/2} + t^{3/2} + t^{5/2} + \cdots.$$

Hence $\tau_j = 0$ for $j \le -3/2$, $\tau_{-1/2} = 1$, $\tau_{1/2} = 0$, and $\tau_j = 1$ for $j \ge 3/2$. The jumps at $n = -1, 0, 1$ are $(1, -1, 1)$; all others vanish.

Exercise 7

The full preimage is $\{(2, 1+3k) : k \in \mathbb{Z}\}$. It is primitive iff $\gcd(2, 1+3k) = 1$, i.e. k is **even** (because $1 + 3k$ is odd exactly when k is even). The open arcs union to $(0.1, 1.2)$ in a chosen lift, an interval of length $1.1 > 1$, and so its mod-one projection is the entire circle; each endpoint is covered by the other arc.

Exercise 8 — model dependency answer

Theorem 1.4 supplies manifold extendability. Propositions 3.8/3.13 (inside Theorem 1.5's proof) reduce to decorated curves. Theorem 2.2 identifies train-track bigons with a box tensor complex under reduced/special-bounded hypotheses. Theorem 4.11 gives the pairing-compatible classification, allowing the manifold geometric pairing Theorem 1.2 after marking invariance. HRW22 Theorems 5 and 7 add gradings and elliptic symmetry. Proposition 62 needs sutured-Floer decategorification and signed rays; Proposition 65 needs knot Floer norm detection; HRW22 Theorem 69 needs Floer simplicity, non-solid-torus-like geometry, and a known interior slope. HRW23 Theorems 7.17 and 1.14 need tangent-slope and pegboard analysis; the gluing theorem requires two incompressible torus boundaries. **This dependency argument is complete as a map, not a reproof of each source's diagrammatic theorem.**

12. Oral examination with model answers

question. Q1. Explain the complete correspondence in two minutes. Answer: The manifold type D module is extendable (HRW23, Thm. 1.4); elementary normal

forms give immersed strands with crossover arrows; the depth algorithm removes finite-depth arrows, leaving parallel-strand monodromy, so every extendable module is represented by decorated curves (Thm. 1.5). Unobstructed test curves and local-system rank tests establish the pairing-compatible bijection (Thm. 4.11). The box tensor and bigon complexes agree for suitable presentations (Thm. 2.2); after marking invariance this yields the manifold pairing (Thm. 1.2). The 2022 companion supplies gradings and elliptic symmetry.

question. Q2. Why is uniqueness not automatic from drawing the curves? Answer: Different train tracks related by basis changes or crossover slides can look different; free homotopy classes and local-system monodromies must be recovered from pairings. HRW23, Lemma 4.14 detects weighted cyclic words with open test tracks, and Proposition 4.15 separates nonsimilar monodromies using parallel test systems.

question. Q3. Why do parallel local systems matter? Answer: For incommensurable primitive circles the Floer dimension is the geometric intersection count multiplied by local-system ranks; monodromy disappears. A parallel test gives a kernel of $I + A \otimes B$, equivalent to an R-module Hom space. Choosing tests for p^ℓ reconstructs the elementary divisors of A.

question. Q4. State the grading formula and its domain of application. Answer: For intersections in the same Spin^c summand and compatible lifts, form a closed decorated path C. Then $m(y) - m(x) = 2N(C) + R(C)/\pi$, with signed puncture winding N and smooth clockwise tangent rotation R excluding intersection-corner jumps. Periodic-domain indeterminacy and phantom grading arrows must be accounted for; there is no canonical absolute grading across summands.

question. Q5. Explain the proof of the torsion formula. Answer: Signed intersections with downward puncture rays and normalized Turaev torsion satisfy the same difference equation under translation by a meridian, because both differences equal the Euler characteristic of sutured Floer homology. Both vanish sufficiently far toward the negative end, so the difference between functions is zero on every translation orbit. This uses HRW22, Lemma 61 and external decategorification input.

question. Q6. How does the Thurston-norm theorem differ from the torsion theorem? Answer: Torsion is an algebraic intersection or decategorification of the oriented curve. The norm theorem uses the highest puncture obstructed from reaching an end in a tight pegboard, and a proper filling-core arc to find the extremal nonzero $\widehat{HFK}$ level. It needs knot Floer norm detection and handles the disk/solid-torus-like case separately.

question. Q7. State L-space gluing without losing a hypothesis. Answer: If compact oriented M_0, M_1 have incompressible torus boundary and are glued by an orientation-reversing h, the result is an L-space iff $\mathcal{L}^\circ(M_0) \cup h(\mathcal{L}^\circ(M_1)) = \mathbb{Q}P^1$. It is a full projective-circle covering condition on *interiors*, not mere overlap. Floer simplicity of both pieces is not assumed in the theorem.

question. Q8. Identify the most dangerous overgeneralization across the two papers. Answer: Treating algebraically extendable decorated curves as if all were manifold-realizable. For example HRW22, Proposition 10 forbids small peripheral-circle components in actual manifold invariants; HRW23, Proposition 7.13 rules out wholly loose curve invariants for boundary-incompressible manifolds. Neither restriction follows merely from the algebraic classification theorem.

13. Reading and source verification

Total study time = 290 minutes (4 hours 50 minutes). Spend 40 min on main-paper §§2–3 (torus algebra, curvature and arrow reduction); 45 min on main-paper §4 (classification and local-system tests); 35 min on main-paper §§5–6 (pairing and Spin^c refinement); 45 min on companion §§2–4 (Maslov, elliptic symmetry, filling cores); 40 min on companion §5 and main-paper §§7.5–7.6 (torsion, norm and L-spaces); 55 min on written exercises; and 30 min on oral rehearsal. During

reading, put a star beside every statement whose proof in these notes is explicitly marked a **source proof sketch**.

Verified main-paper references: J. Hanselman, J. Rasmussen, L. Watson, *Bordered Floer homology for manifolds with torus boundary via immersed curves*, arXiv:1604.03466v3, dated 25 April 2023 (journal version: *J. Amer. Math. Soc.* **37** (2024), 391–498, DOI 10.1090/jams/1029). Theorem 2.2 appears in §2.6 (printed p. 13); Theorem 4.11 in §4.2 (p. 46); Proposition 4.15 in §4.3 (p. 52); Theorem 7.15 in §7.3 (p. 71); Theorem 7.17 in §7.5 (p. 78); the proof of Theorem 1.14 is in §7.6 (pp. 80–82). These are the **arXiv-v3 printed pages**, not the journal pagination. Main paper — arXiv v3 · Journal DOI.

Verified companion references: J. Hanselman, J. Rasmussen, L. Watson, *Heegaard Floer homology for manifolds with torus boundary: properties and examples*, *Proc. London Math. Soc.* **125** (2022), 879–967, DOI 10.1112/plms.12473. Published numbering: Theorem 5 (relative grading), Theorem 7 (elliptic symmetry), Theorem 11 (knot-complement curve from CFK^-), Proposition 12 (filling-core Floer pairing), Theorem 27 (HF solid tori, under its manifold hypotheses), Definition 29 (grading arrows), Lemma 61 and Proposition 62 (torsion), Proposition 65 (norm), Lemma 66, Proposition 67, Lemma 68, Theorem 69 (L-space interval). Companion — published open-access HTML · Companion DOI.

Supporting only: Lipshitz–Ozsváth–Thurston, *Bordered Heegaard Floer Homology*, *Mem. Amer. Math. Soc.* **254** (2018), no. 1216, DOI 10.1090/memo/1216. The original algebraic pairing, graded group and knot-complement algorithms are treated as external inputs, not the central reading today.

Next examination: HRW Day 21 — timed qualifying-exam simulation, counterexample defense, and final synthesis.