HRW Day 21 — Final Qualifying Examination: Proof Dependencies, Computations, and Oral Defense

2026-11-12

Day 21 — Final Qualifying Examination: Proof Dependencies, Computations, and Oral Defense

Course date: 2026-11-12. **Language:** English. **Target:** graduate qualifying examination. **Study allocation:** 4 h 45 min.

PDF attachment: [[HRW_Day21.pdf]]. **Source:** [[HRW_Day21_Source.tex]] (same folder). These are newly authored replacement notes for previously unavailable Day 21 files, not recovered copies of the original lectures.

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Reading assignment and theorem scope

Main paper: HRW23 §§2–7, emphasizing Theorems 1.4, 1.5, 2.2, 4.11, 1.14 and 7.17; HRW22 §§2–5, emphasizing Theorems 5 and 7, Propositions 62 and 65, and Theorem 69. Consult HRW23 and HRW22. These notes follow the **source's algebraic and geometric terminology**. A statement identified as a source theorem is not claimed to have been proved in full merely because an illustrative calculation is supplied. We use the 2023 main manuscript and the 2022 published companion; theorem numbers can differ in earlier preprints.

Core definitions and exact algebraic framework

> [!theorem] Source-grounded mathematical content This final session is a closed-book reconstruction of the two HRW papers. The main chain is: manifold extendability (HRW23 Theorem 1.4), train-track normal form and crossover-arrow depth reduction (Propositions 3.8 and 3.13), existence of decorated curves (Theorem 1.5), uniqueness via test pairings (Theorem 4.11), chain-level box tensor/bigon correspondence (Theorem 2.2), and the manifold pairing theorem (Theorem 1.2). The companion supplies the graded pairing (Theorem 5), elliptic symmetry (Theorem 7), manifold restrictions (Proposition 10), Turaev torsion from ray intersections (Proposition 62), Thurston-norm detection (Proposition 65), and torsion-based interior L-space intervals (Theorem 69). The geometric L-space gluing criterion (HRW23 Theorem 1.14) requires boundary incompressibility and full coverage of the rational slope circle by the two transported **interior** L-space sets.

Context and motivation. Bordered Floer theory expresses gluing through a type A/type D pairing, while the immersed-curve reformulation turns the pairing into a geometric intersection problem on a once-punctured torus. The technical challenge is preserving the information of the coefficient algebra, local systems, and admissibility under graphical changes. A concrete computation must therefore be separated from a theorem asserting equivalence of whole modules.

Categories. The geometric model is a punctured **surface** with immersed curves and finite-dimensional local systems. It does not assert that every algebraically admissible curve system comes from a smooth three-manifold. All results here are over $\mathbb{F}_2$.

The main proof or proof checkpoint

> [!proof] Mathematical argument and its status **Proof A, complete: local-system separation.** Set $R = \mathbb{F}_2[t, t^{-1}]$. The PID decomposition of an invertible monodromy A has blocks $R/(p^{a_j})$. For an irreducible p of degree d, $\dim \mathrm{Hom}_R(R/(p^\ell), R/(p^a)) = d \min(\ell, a)$ because a map is determined by an element annihilated by p^ℓ. A parallel test with inverse-transpose companion matrix $B_{p,\ell}$ computes $D_p(\ell; A) = d \sum_j \min(\ell, a_j)$. The first discrete difference counts blocks of length at least ℓ, and the second recovers blocks of exact length ℓ. Thus these tests recover the similarity class. **Proof B, complete conditional finite-difference step:** if signed downward ray intersections n_s and normalized torsion τ_s have equal meridional differences and both vanish sufficiently far in the negative direction, then $n_s - \tau_s$ is constant and zero. The equality of differences uses the filling-core/sutured Floer inputs of HRW22 Proposition 62. **Proof C, source-dependent:** the L-space gluing theorem's two contrapositives use common-chord tangency, pegboard limits, and persistent opposite-parity intersections; the complete local cases require HRW23 Figures 62–63.

What to defend orally. Identify the exact algebraic equation, explain why its hypotheses hold, and distinguish the complete calculation above from the source-dependent geometric normal-form or bigon argument. The latter requires the relevant diagrams of HRW23; the text here does not silently replace those diagrams with a heuristic.

Worked computation and interpretation

Example 1 (homology and intersection). For primitive $u = (2, 3)$ and $v = (1, 4)$, $|\det(u, v)| = 5$. Rank-two and rank-three systems on straight nonparallel curves yield 30 Floer generators and no bigon differential. For an integral-homology-solid-torus gluing with row matrix $H = \left(\begin{smallmatrix} 3 & 5 \\ -1 & -2 \end{smallmatrix}\right)$, $\det H = -1$ and Mayer–Vietoris gives $H_1(Y) \cong \mathbb{Z}/5$. This does not establish the L-space property. **Example 2 (trefoil torsion).** $\Delta_{T(2,3)}(t) = t^{-1} - 1 + t$ and $t^{1/2}\Delta(t)/(1-t) = t^{-1/2} + t^{3/2} + t^{5/2} + \cdots$. The nonzero jumps are $1, -1, 1$ at gradings $-1, 0, 1$. **Example 3 (relative Maslov).** For a right-angle bigon with smooth rotation $-\pi$, $\Delta m = 2N - 1$: -1 when puncture-free and $+1$ with one positive puncture winding. The punctured domain is not counted by the hat differential.

Second check (algebraic invariant). An invertible matrix A and a basis change P define isomorphic local systems precisely when the monodromies are conjugate, $A' = P^{-1}AP$. Similarity preserves characteristic and minimal polynomials, but agreement of those two polynomials alone need not imply similarity. This observation becomes decisive in the classification proof.

Geometric check. For primitive nonparallel slope vectors $u = (a, b)$ and $v = (c, d)$, $i(u, v) = |ad - bc|$. This formula counts geometric intersections of straight representatives; when the curves are parallel, or when local-system monodromy is nontrivial, the Floer homology cannot be read from the determinant alone.

Source-specific pitfalls

> [!warning] Convention and hypothesis audit Never claim every extendable type D module is manifold-realizable. Never identify a phantom grading arrow with a mon-

odromy crossover arrow. Never treat minimal geometric intersection as Floer rank in the commensurable case. Never replace a complete boundary preimage $j_*^{-1}(D^\tau)$ by one chosen lift. Never use the L-space gluing theorem without boundary incompressibility or replace union coverage by interval overlap. Theorem 69 determines the correct **interior** interval given a reference slope, not all endpoint membership. The rational pinch inequality is not a theorem for arbitrary degree-one maps.

For any argument involving closed-manifold Floer homology, the gluing homeomorphism reverses boundary orientation. For the type A/type D pairing, the elliptic-involution correction and reflection convention must be tracked; HRW22's elliptic symmetry permits simplification for **actual manifold invariants**, not automatically for every abstract curve collection.

Written qualifying examination

Instructions. Attempt these six questions without notes; write complete algebraic proofs and explicitly label any diagrammatic proof sketch.

[!exercise] Problem 1 Reconstruct the type D equation and curvature extension, giving one explicit matrix calculation.

[!exercise] Problem 2 Explain the depth algorithm and identify the one diagrammatic step you cannot prove without HRW23 §3.6.

[!exercise] Problem 3 Prove the elementary-divisor local-system separation lemma completely.

[!exercise] Problem 4 Calculate a nonparallel rank-two/rank-three immersed Floer pairing and a parallel rank-one pairing.

[!exercise] Problem 5 Prove the finite-difference uniqueness argument for normalized Turaev torsion, specifying its external Floer inputs.

[!exercise] Problem 6 State HRW22 Theorem 69 and compute a full boundary preimage under a map to $\mathbb{Z} \oplus \mathbb{Z}/3$.

[!exercise] Problem 7 State L-space gluing with exact hypotheses, then give both contrapositives as explicitly labelled geometric proof sketches.

[!exercise] Problem 8 Explain why rank five for essential separating tori, the rational pinch inequality, and the satellite necessary condition are not consequences of determinant arithmetic alone.

Solutions and hints

[!proof] Problem 1: solution strategy / checkpoint Write the definitions first and check every algebra product and idempotent. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 2: solution strategy / checkpoint Use the explicitly displayed equations and distinguish identities from existence theorems. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 3: solution strategy / checkpoint Reduce to a finite-dimensional linear-algebra calculation or a chain-level coefficient check. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 4: solution strategy / checkpoint Keep the geometric representative in minimal position; for parallel curves use the intertwiner complex. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 5: solution strategy / checkpoint Cite the exact source lemma for the nontrivial geometric step; do not assert it follows from a determinant. **Answer**

source: the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 6: solution strategy / checkpoint State the category and hypothesis restrictions and give a counterexample to the stronger assertion. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 7: solution strategy / checkpoint Write the definitions first and check every algebra product and idempotent. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

[!proof] Problem 8: solution strategy / checkpoint Use the explicitly displayed equations and distinguish identities from existence theorems. **Answer source:** the core proof and worked computation above provide the necessary equations. A complete diagrammatic proof, where required, must use the assigned HRW figures.

Oral examination and model answers

[!question] Question 1: What is the central result of this lecture? **Model answer:** This final session is a closed-book reconstruction of the two HRW papers.

[!question] Question 2: Which calculation here is proved completely? **Model answer:** **Proof A, complete: local-system separation.

[!question] Question 3: What is the most important convention pitfall? **Model answer:** Never claim every extendable type D module is manifold-realizable.

[!question] Question 4: Why is a determinant insufficient in general? **Model answer:** A determinant counts minimal intersections of straight slope representatives, not the full Floer differential or the monodromy of parallel local systems.

[!question] Question 5: Which argument needs a source diagram? **Model answer:** The train-track, crossover-arrow or immersed-bigon equivalence step identified explicitly as a source-dependent proof sketch.

[!question] Question 6: How does the companion paper enter? **Model answer:** It supplies gradings, elliptic symmetry and manifold-specific restrictions beyond the abstract ungraded classification; the exact use depends on the assigned section.

Time plan and final verification

Segment	Minutes	Goal
Source close reading	75	Check exact statements and diagrams
Definitions and conventions	35	Restate hypotheses without notes
Proof reconstruction	60	Separate full proofs and sketches
Worked calculations	40	Check algebra independently
Six written problems	50	Produce complete or qualified answers
Oral defense	25	Six model questions

Total: 285 minutes = 4 h 45 min.

Source references: HRW23 (especially HRW23 §§2–7, emphasizing Theorems 1.4, 1.5, 2.2, 4.11, 1.14 and 7.17; HRW22 §§2–5, emphasizing Theorems 5 and 7, Propositions 62 and 65, and Theorem 69.); HRW22. Supporting LOT memoir is background only.